Intermediate Microeconomics

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Problem Set 6: suggested solutions

1. Consider the following two player game:

player 2

player 1		a	b	c	d
	A	$3,6$	$4,7$	$2,5$	$0,6$
	B	$7,2$	$5,3$	$3,0$	$3,2$
	C	$4,3$	$4,2$	$2,1$	$1,6$

(a) Does any player have a strictly dominant strategy? Find the strictly dominant strategies solution, if any.

Solution: For player 1, B is strictly dominant. Player 2 does not have a strictly dominant strategy since b is best against A but d is best against C. There is no sDSS.

(b) Find the iterated elimination of strictly dominated strategies solution, if any.

Solution: Eliminating A, C, a, c, and d in this order yields (B, b) as the IEsDSS.

(c) Find all the Nash equilibria, if any.

Solution: (B, b) is also the unique Nash Equilibrium.

2. Consider the following model of Cournot competition with differentiated goods. Two firms compete by setting quantities. They have the identical cost function $c_i(q_i) = cq_i$. However, they face different (inverse) demand:

$$p_i(q_i, q_j) = (a - bq_i - dq_j), \quad \text{where } a > c \text{ and } i \neq j.$$

(a) Find the firms' best response functions.

Solution: Let $i \neq j$. Since $p_i = a - bq_i - dq_j$, we have

$$\pi_i(q_i, q_j) = (a - bq_i - dq_j)\, q_i - cq_i$$

To find $BR_i(q_j)$, we differentiate π_i with respect to q_i:

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = (a - bq_i - dq_j) - bq_i - c = 0$$
$$\implies a - cdq_j = 2bq_i$$
$$q_i = \frac{a-c}{2b} - \frac{d}{2b}q_j$$

To complete the answer, we note that

$$q_i \geq 0 \iff \frac{a-c}{2b} - \frac{d}{2b}q_j \iff \frac{a-c}{d} \geq q_j.$$

Putting everything together, we have

$$BR_i(q_j) = \begin{cases} \frac{a-c}{2b} - \frac{d}{2b}q_j & \text{if } q_j \leq \frac{a-c}{d} \\ 0 & \text{if } q_j > \frac{a-c}{d}. \end{cases}$$

(b) Find the Nash equilibrium output levels.

Solution: To find the Nash equilibrium, we solve $q_1 = BR_1(q_2)$ and $q_2 = BR_2(q_1)$. That is, assuming positive production,

$$q_1 = \frac{a-c}{2b} - \frac{d}{2b}q_2 \quad \text{and} \quad q_2 = \frac{a-c}{2b} - \frac{d}{2b}q_1$$

$$\implies q_1 = \frac{a-c}{2b} - \frac{d}{2b}\left(\frac{a-c}{2b} - \frac{d}{2b}q_1\right)$$
$$\implies q_1 = \frac{a-c}{2b} - \frac{(a-c)d}{4b^2} + \frac{d^2 q_1}{4b^2}$$
$$\implies \left(1 - \frac{d^2}{4b^2}\right) q_1 = \left(\frac{a-c}{2b}\right)\left(1 - \frac{d}{2b}\right)$$
$$\implies \left(1 - \frac{d}{2b}\right)\left(1 + \frac{d}{2b}\right) q_1 = \left(\frac{a-c}{2b}\right)\left(1 - \frac{d}{2b}\right)$$
$$q_1 = \left(\frac{a-c}{2b}\right)\left(\frac{2b}{2b+d}\right)$$
$$q_1^{\text{NE}} = \frac{a-c}{2b+d} \implies q_2^{\text{NE}} = \frac{a-c}{2b+d} \quad \text{by same argument.}$$

(c) Suppose the government imposes a license fee so that any firm operating in this market (that is, produces a positive amount output) must pay a fixed (lump-sum) license fee F. Suppose $a = 100$, $b = 1$, $d = 2$, and $c = 1$. If $F = 10$, how does this affect the Nash equilibrium output levels of the firms? How about the profits of the firms?

Solution: Since fixed fee is lump-sum, it does not change the best response functions or the equilibrium of the game, provided that both firms still produce. Assuming that this is the case, part (b) yields,

$$q_1^{\text{NE}} = q_2^{\text{NE}} = \frac{a-c}{2b+d} = \frac{100-1}{2(1)+2} = \frac{99}{4}.$$
$$\implies p_1^{\text{NE}} = p_2^{\text{NE}} = 100 - \left(\frac{99}{4}\right) - 2\left(\frac{99}{4}\right) = \frac{400-297}{4} = \frac{103}{4}$$
$$\implies \pi_1^{\text{NE}} = \pi_1^{\text{NE}} = \left(\frac{103}{4}\right)\left(\frac{99}{4}\right) - (1)\left(\frac{99}{4}\right) - \text{fee}$$
$$= \left(\frac{99}{4}\right)\left(\frac{99}{4}\right) - \text{fee} = 612.5625 - 10 = 602.5625.$$

Since both firms are still making positive profit even with the license fee, this is the equilibrium output and profit.

(d) Is there a level of license fee at which one of the firms will drop out of the market?

Solution: We can see from part (c) that when the license fee is greater than 612.5625, at lease one of the firm will drop out. If exactly one firm drop outs, the market results in a monopoly. If the fee is large enough, both firms will drop out.

3. [Not to be graded] One of the results discussed in the lecture is:

Theorem: Suppose strategy s_i' (in a finite game) is strictly dominated by strategy $\hat{s}_i$ when opponents are restricted to using pure strategies. That is,

$$\pi_i(\hat{s}_i, s_{-i}) \;>\; \pi_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

Then strategy s_i' will never be in the support of a best response. That is, for all σ_{-i},

$$\sigma_i \in BR_i(\sigma_{-i}) \implies \sigma_i(s_i') = 0.$$

□

(a) Use this result to find all the Nash equilibria (including mixed strategy ones) of the following game.

player 1 \ player 2	L	R
T	$10, 5$	$4, 1$
M	$6, 1$	$8, 8$
B	$5, 0$	$0, 10$

Solution: Since B is strictly dominated by T, it will never be in the support of any best response. Therefore, we can reduce the game to

player 1 \ player 2	L	R
T	$10, 5$	$4, 1$
M	$6, 1$	$8, 8$

The pure strategy Nash equlbria of this game are (T, L), (M, R). To look for the mixed strategy Nash equilibrium of the form $((p, 1-p), (q, 1-q))$, we solve

$$\begin{aligned}
(1)\ \pi_1(T, (q, 1-q)) &= \pi_1(M, (q, 1-q)) \\
10q + 4(1-q) &= 6q + 8(1-q) \\
4 + 6q &= 8 - 2q \\
\implies q &= \tfrac{4}{8} = \tfrac{1}{2}. \\
(2)\ \pi_2((p, 1-p), L) &= \pi_2((p, 1-p), R) \\
5p + 1(1-p) &= 1p + 8(1-p) \\
1 + 4p &= 8 - 7p \\
\implies p &= \tfrac{7}{11}.
\end{aligned}$$

So the mixed strategy Nash equilibrium is $((\frac{7}{11}, \frac{4}{11}, 0), (\frac{1}{2}, \frac{1}{2}))$, where 0 is the weight placed on B.

(b) The above theorem holds even when s_i is strictly dominated by a mixed strategy. That is, when there is some $\hat{\sigma}_i$ such that

$$\pi_i(\hat{\sigma}_i, s_{-i}) > \pi_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

Use this result to find all the Nash equilibria (including mixed strategy ones) of the following game.

		player 2	
		L	R
player 1	T	8, 1	0, 0
	M	4, 0	12, 2
	B	6, 0	5, 10

Solution Payoff from B lies between the payoffs from T and M no matter what s_2 is. This suggests that there may be $\sigma_1' = (\alpha, 1-\alpha, 0)$ that strictly dominates B. To see whether such σ_1' really exists, we solve

$$\begin{aligned}
\alpha(8) + (1-\alpha)4 > 6 &\implies \alpha > \tfrac{1}{2} = \frac{12}{24} \\
\alpha(0) + (1-\alpha)12 > 5 &\implies \alpha < \tfrac{7}{12} = \frac{14}{24}.
\end{aligned}$$

So $\sigma_1' = (\frac{13}{24}, \frac{11}{24}, 0)$ strictly dominates B. So once again we can reduce the game to

		player 2	
		L	R
player 1	T	8, 1	0, 0
	M	4, 0	12, 2

The pure strategy Nash equilibria are (T, L) and (B, R). To look for the mixed strategy Nash equilibrium of the form $((p, 1-p), (q, 1-q))$, we solve

$$
\begin{aligned}
(1)\ \pi_1(T,(q,1-q)) &= \pi_1(M,(q,1-q)) \\
8q+0(1-q) &= 4q+12(1-q) \\
8q &= 12-8q \\
\Longrightarrow q &= \tfrac{12}{16} = \tfrac{3}{4}. \\
(2)\ \pi_2((p,1-p),L) &= \pi_2((p,1-p),R) \\
1p &= 2(1-p) \\
p &= 2-2p \\
\Longrightarrow p &= \tfrac{2}{3}.
\end{aligned}
$$

So the mixed strategy Nash equilibrium is $((\frac{2}{3},\frac{1}{3},0),(\frac{3}{4},\frac{1}{4}))$, where 0 is the weight placed on B.