Intermediate Microeconomics

Fall 2024 - M. Chen, M. Pak, and B. Xu

Problem Set 5: suggested solutions

1. Consider an Edgeworth box economy where preferences are given by

$$u^1(x_1^1, x_2^1) = 2\sqrt{x_1^1} + x_2^1 \quad \text{and} \quad u^2(x_1^2, x_2^2) = 2\sqrt{x_1^2} + x_2^2,$$

(a) Suppose the initial endowments are $e^1 = (4,4)$ and $e^2 = (1,1)$. Find all the Pareto optimal allocations.

Solution: Interior Pareto optimality is characterized by marginal rates of substitution being equal to each other. Thus,

$$MRS^1 = \frac{\frac{\partial u^1}{\partial x_1^1}}{\frac{\partial u^1}{\partial x_2^1}} = \frac{1}{\sqrt{x_1^1}} = \frac{1}{\sqrt{x_1^2}} = \frac{\frac{\partial u^2}{\partial x_1^2}}{\frac{\partial u^2}{\partial x_2^2}} = MRS^2.$$

Thus, $x_1^1 = x_1^2$. Since $x_1^1 + x_1^2 = \bar{e}_1 = 5$, interior PO allocations are characterized by $x_1^1 = x_1^2 = 2.5$. So

$$\text{Interior PO set } = \left\{\left((2.5, x_2^1), (2.5, 5 - x_2^1)\right) : 0 \le x_2^1 \le 5\right\}.$$

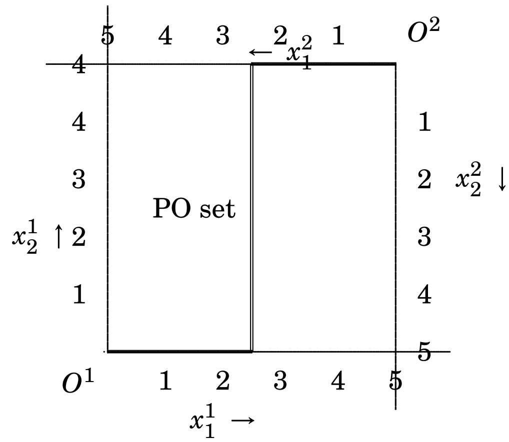
Note that when $x_1^1 < 2.5$ and $x_2^1 = 0$ (which means $x_1^2 > 2.5$ and $x_2^2 = 5$), we have

$$MRS^1 = \frac{1}{\sqrt{x_1^1}} > \frac{1}{\sqrt{x_1^2}} = MRS^2.$$

So this is a boundary PO allocation. Similarly, when $x_1^1 > 2.5$ and $x_2^1 = 5$ (which means $x_1^2 < 2.5$ and $x_2^2 = 0$), we have

$$MRS^1 = \frac{1}{\sqrt{x_1^1}} < \frac{1}{\sqrt{x_1^2}} = MRS^2.$$

So this is a boundary PO allocations. The set of all the PO allocations are graphed below:



(b) Find the competitive equilibrium (also called Walrasian equilibrium).

Solution: To find the Walrasian equilibrium, we first find the demand functions. Using normalization $p_2 = 1$,

$$MRS^1 = \frac{1}{\sqrt{x_1^1}} = \frac{p_1}{1} \implies x_1^1 = \frac{1}{p_1^2}$$

$$MRS^2 = \frac{1}{\sqrt{x_1^2}} = \frac{p_1}{1} \implies x_1^2 = \frac{1}{p_1^2}.$$

Market clearing condition for Market 1 yields

$$x_1^1 + x_1^2 = \bar{e}_1 \implies \frac{1}{p_1^2} + \frac{1}{p_1^2} = 5 \implies p_1^* = \sqrt{\frac{2}{5}} = 0.6325$$

$$x_1^{1^*} = \frac{1}{\frac{2}{5}} = 2.5 \quad \text{and} \quad x_1^{2^*} = \frac{1}{\frac{2}{5}} = 2.5$$

To find equilibrium amount of good 2, we use

$$\begin{aligned}
p_1 x_1^1 + p_2 x_2^1 = 4p_1 + 4p_2 &\implies 0.6325(2.5) + x_2^1 = 4(0.6325) + 4 \\
&\implies x_2^{1^*} = 4 + 1.5(0.6325) = 4 + 0.9488 = 4.9488 \\
p_1 x_1^2 + p_2 x_2^2 = 1p_1 + 1p_2 &\implies 0.6325(2.5) + x_2^2 = 1(0.6325) + 1 \\
&\implies x_2^{2^*} = 1 - 1.5(0.6325) = 1 - 0.9488 = 0.0512.
\end{aligned}$$

To recap, the equilibrium prices are $p^* = (0.6325, 1)$ and the equilibrium allocation is $\left(x^{1^*}, x^{2^*}\right) = \left((2.5, 4.9488), (2.5, 0.0512)\right)$.

2. Consider an Edgeworth box economy where preferences are given by

$$u^1(x_1^1, x_2^1) = x_1^1 + \ln x_2^1 \quad \text{and} \quad u^2(x_1^2, x_2^2) = x_1^2 x_2^2,$$

and the initial endowments are

$$e^1 = (1,3) \quad \text{and} \quad e^2 = (3,1).$$

Find all the competitive (Walrasian) equilibrium. You may assume that the solution is interior.

Solution: To find Consumer 1's Marshallian demand, we solve

$$\max_{x_1^1, x_2^1} x_1^1 + \ln x_2^1 \quad \text{s.t.} \quad p_1 x_1^1 + p_2 x_2^1 = p_1 + 3p_2$$

Setting MRS = price ratio yields

$$x_2^1 = \frac{p_1}{p_2}.$$

Substituting this into (3) yields:

$$p_1 x_1^1 + p_2\left(\frac{p_1}{p_2}\right) = p_1 + 3p_2 \quad \Rightarrow \quad x_1^1 = \frac{3p_2}{p_1}.$$

Note that this also shows that there are no boundary solutions in this case. So, Marshallian demand for consumer 1 is:

$$x^1(p_1, p_2) = \left(\frac{3p_2}{p_1}, \frac{p_1}{p_2}\right).$$

To find consumer 2's Marshallian demand function, we note that her utility function is a Cobb-Douglas utility function, so the demand function is given by

$$x^2(p_1, p_2) = \left(\frac{3p_1 + p_2}{2p_1}, \frac{3p_1 + p_2}{2p_2}\right).$$

Now, we find the equilibrium prices by normalizing $p_2 = 1$ and looking for market clearing prices for the market for good 2:

$$x_2^1(p_1, 1) + x_2^2(p_1, 1) = e_2^1 + e_2^2 \quad \Rightarrow \quad \frac{p_1}{1} + \frac{3p_1 + 1}{2} = 4$$
$$\Rightarrow \quad p_1 = \frac{7}{5}.$$

So, the Walrasian equilibrium price vector is $p = (\frac{7}{5}, 1)$, and the corresponding Walrasian equilibrium allocation is:

$$x^1 = \left(\frac{15}{7}, \frac{7}{5}\right), \quad \text{and} \quad x^2 = \left(\frac{13}{7}, \frac{13}{5}\right).$$

3. Consider an Edgeworth box economy with two consumers, whose utility functions and endowments are

$$u^1(x_1^1, x_2^1) = (x_1^1)(x_2^1)^{\frac{1}{3}} \qquad e^1 = (10,0)$$
$$u^2(x_1^2, x_2^2) = (x_1^2)(x_2^2)^{\frac{1}{4}} \qquad e^2 = (0,10)$$

In the following, use the normalization $p_2 = 1$.

(a) Find the competitive (Walrasian) equilibrium.

Solution: We first transform the utilities into a standard Cobb-Douglas ones:

$$
\begin{aligned}
u^1(x_1^1, x_2^1) &= (x_1^1)(x_2^1)^{\frac{1}{3}} \sim (x_1^1)^{\frac{3}{4}}(x_1^1)^{\frac{1}{4}} \\
u^2(x_1^2, x_2^2) &= (x_1^2)(x_2^2)^{\frac{1}{4}} \sim (x_1^2)^{\frac{4}{5}}(x_2^2)^{\frac{1}{5}}.
\end{aligned}
$$

The incomes of the two consumers are

$$
\begin{aligned}
I_1 &= 10p_1 \\
I_2 &= 10p_2 = 10
\end{aligned}
$$

The Marshallian demand functions are

$$
\begin{aligned}
x_1^1(p_1, p_2) &= \frac{3}{4}\frac{10p_1}{p_1} = \frac{30}{4} \\
x_2^1(p_1, p_2) &= \frac{1}{4}\frac{10p_1}{p_2} = \frac{10p_1}{4} \\
x_1^2(p_1, p_2) &= \frac{4}{5}\frac{10}{p_1} = \frac{40}{5p_1} \\
x_2^2(p_1, p_2) &= \frac{1}{5}\frac{10}{p_2} = \frac{10}{5} = 2.
\end{aligned}
$$

We'll seek to clear the second market:

$$
\begin{aligned}
x_2^1(p_1, p_2) + x_2^2(p_1, p_2) &= \frac{10p_1}{4} + 2 = 10 = e_2^1 + e_2^2 \\
10p_1 &= 32 \implies \hat{p}_1 = \frac{32}{10} = \frac{16}{5} = 3.2.
\end{aligned}
$$

Equilibrium allocations are

$$
\begin{aligned}
\hat{x}_1^1 &= \tfrac{30}{4} = \tfrac{15}{2} = 7.5 \\
\hat{x}_2^1 &= \tfrac{10}{4}\left(\tfrac{16}{5}\right) = \tfrac{2(4)}{1} = 8 \\
\hat{x}_1^2 &= \tfrac{40}{5}\left(\tfrac{5}{16}\right) = \tfrac{5}{2} = 2.5 \\
\hat{x}_2^2 &= 2.
\end{aligned}
$$

(b) State the first fundamental theorem of welfare and verify that it holds in this economy.

Solution: The first fundamental theorem of welfare states that under a mild set of conditions (preferences are monotone), every competitive equilibrium is Pareto optimal. In the current example, at the competitive equilibrium, we have

$$
\begin{aligned}
MRS^1 &= \frac{(x_2^1)^{\frac{1}{3}}}{\frac{1}{3}(x_1^1)(x_2^1)^{-\frac{2}{3}}} = \frac{3(x_2^1)}{(x_1^1)} = \frac{(3)8}{7.5} = \frac{(3)(8)(2)}{15} = \frac{16}{5} = 3.2 \\
MRS^2 &= \frac{(x_2^2)^{\frac{1}{4}}}{\frac{1}{4}(x_1^2)(x_2^2)^{-\frac{4}{5}}} = \frac{4(x_2^2)}{(x_1^2)} = \frac{4(2)}{2.5} = \frac{4(2)(2)}{5} = \frac{16}{5} = 3.2.
\end{aligned}
$$

So, it is indeed Pareto optimal.

(c) Consider the allocation

$$\widetilde{x} = (\widetilde{x}^1, \widetilde{x}^2) = \big((2,2.5),(8,7.5)\big).$$

Can this allocation be supported as an equilibrium with transfers? If so, find the transfers and prices that will make this allocation an equilibrium with transfers.

Solution: At $\widetilde{x}$,

$$MRS^1 = \frac{3(x_2^1)}{(x_1^1)} = \frac{(3)2.5}{2} = 3.75.$$
$$MRS^2 = \frac{4(x_2^2)}{(x_1^2)} = \frac{4(7.5)}{8} = 3.75.$$

So, $\widetilde{x}$ can be supported as an equilibrium with transfer. Since the slope of the indifference curves and the budget lines have to be tangent, we need

$$\frac{\widetilde{p}_1}{\widetilde{p}_2} = \widetilde{p}_1 = 3.75.$$

So the supporting prices vector is $\widetilde{p} = (3.75, 1)$. The transfers are

$$\begin{aligned} T_1 &= \widetilde{p}_1\widetilde{x}_1^1 + \widetilde{p}_2\widetilde{x}_2^1 - \widetilde{p}_1 e_1^1 + \widetilde{p}_2 e_2^1 \\ &= 3.75(2) + 1(2.5) - 3.75(10) - 1(0) = 10 - 37.5 = -27.5 \\ T_2 &= \widetilde{p}_1\widetilde{x}_1^2 + \widetilde{p}_2\widetilde{x}_2^2 - \widetilde{p}_1 e_1^2 + \widetilde{p}_2 e_2^2 \\ &= 3.75(8) + 1(7.5) - 3.75(0) - 1(10) = 37.5 - 10 = +27.5. \end{aligned}$$

4. A monopolist with cost function $c(Q) = \frac{Q^2}{2a}$ faces an inverse demand function given by $P(Q) = \frac{2}{\sqrt{Q}}$.

(a) Find the elasticity of demand with respect to price.

Solution:

$$\epsilon_D = \left(\frac{dQ}{dP}\right)\left(\frac{P}{Q}\right) = \left(\left(\frac{dP}{dQ}\right)\left(\frac{Q}{P}\right)\right)^{-1} = \left(-Q^{-\frac{3}{2}}\frac{Q}{2Q^{-\frac{1}{2}}}\right)^{-1} = -2.$$

(b) Assuming that the monopolist uses $MR = MC$ pricing rule, find his profit maximizing price, p^m, and output level, q^m.

Solution: The monopolist solves

$$\max_Q P(Q)Q - C(Q) \iff \max_Q 2Q^{\frac{1}{2}} - \frac{Q^2}{2a}.$$

$$\begin{aligned} FOC \Rightarrow \quad Q^{-\frac{1}{2}} - \frac{1}{a}Q &= 0 \\ 1 - \frac{1}{a}Q^{\frac{3}{2}} &= 0 \\ Q^m &= a^{\frac{2}{3}} \\ P^m &= \frac{2}{\sqrt{Q^m}} = \frac{2}{a^{\frac{1}{3}}} \end{aligned}$$

(c) Find the marginal cost at q^m and calculate the Lerner index.

Solution:

$$\begin{aligned} MC(q^m) &= \left(\frac{1}{a}\right) a^{\frac{2}{3}} = \frac{1}{a^{\frac{1}{3}}} \\ LI &= \frac{P^m - MC}{P^m} = \frac{\frac{2}{a^{\frac{1}{3}}} - \frac{1}{a^{\frac{1}{3}}}}{\frac{2}{a^{\frac{1}{3}}}} = \frac{1}{2}. \end{aligned}$$

(d) Does the monopolist's market power depend on his cost curve? In particular, does it depend on a? Is your answer surprising?

Solution: No. Since the price elasticity of demand is constant along the demand curve, we would expect Lerner index to be constant as well:

$$LI = \frac{1}{|\varepsilon_D|} = \frac{1}{2}.$$

5. A monopolist with zero cost, that is $c(q) = 0$, faces two consumers whose demand functions are given below.

$$\begin{aligned} Q_1 &= 6 - P \\ Q_2 &= 12 - 2P \end{aligned}$$

(a) Suppose the monopolist cannot engage in any price discrimination. Find the firm's optimal pricing strategy. Calculate the firm's Lerner index.

Solution: The aggregate demand and the inverse aggregate demand is give by

$$Q = Q_1 + Q_2 = 18 - 3P \quad \text{and } P = 6 - \frac{Q}{3}.$$

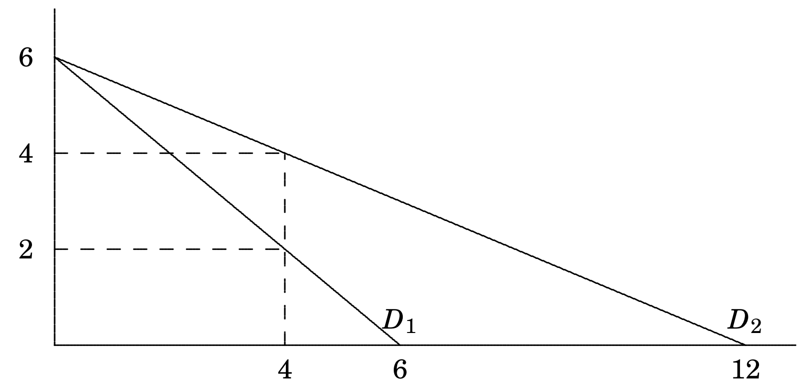
The optimal pricing strategy is found by setting $MR = MC$, which yields

$$\begin{aligned} MR = \frac{d}{dQ}\left[\left(6 - \frac{Q}{3}\right)Q\right] &= 0 = MC \\ 6 - \frac{2Q}{3} &= 0 \\ Q^m &= 6\left(\frac{3}{2}\right) = 9 \\ P^m &= 6 - \frac{9}{3} = 3. \end{aligned}$$

The Learner index is $LI = \frac{P-MC}{P} = \frac{3-0}{3} = 1$.

(b) Now, assume that price discrimination is possible. Find the monopolist's optimal first degree price-discrimination strategy.

Solution:



Generally, first degree price-discrimination strategy involves the firm selling each unit of the good to the consumer who values it most at exactly the consumer's valuation, thereby extracting all the surpluses from the consumers. Here, the marginal cost is constant, so the monopolist can simply offer the quantity - total price pair that will extract all the surpluses to each consumer.

The total surplus available to the consumers are

$$\begin{aligned} CS_1 &= (0.5)(6)(6) = 18, \quad \text{and} \\ CS_2 &= (0.5)(6)(12) = 36. \end{aligned}$$

So, the firm should offer $(Q, TP) = (6, 18)$ to consumer 1 and $(Q, TP) = (12, 36)$ to consumer 2.

(c) Find the monopolist's optimal second degree price-discrimination strategy.

Solution: To find the quantity to be offered to consumer, we set

$$\begin{aligned} 2P_1 &= P_2 \\ 2(6-Q) &= 6 - \frac{1}{2}(Q) \\ 24 - 4Q &= 12 - Q \\ \Rightarrow \quad Q &= 4 \end{aligned}$$

The total surplus available to consumer 1 at $Q = 4$ is $0.5(6-2)(4) + (2)(4) = 16$. So the monopolist should offer $(Q_1, TP_1) = (4, 16)$ in the menu. If consumer 2 takes the option $(4, 16)$ her total surplus will be $0.5(6-4)(4) + (4-2)(4) - 0.5(6-2)(4) = 4 + 8 - 8 = 4$. So, the monopolist should offer $(Q_2, TP_2) = (12, 36-4) = (12, 32)$ as the second option in the menu.

(d) Find the monopolist's optimal third degree price-discrimination strategy.

Solution: The inverse demand functions for the two consumers are

$$\begin{aligned} P_1 &= 6 - Q_1 \\ P_2 &= 6 - \tfrac{1}{2}Q_2. \end{aligned}$$

In third-degree price discrimination, the monopolist solves

$$\begin{aligned}
& \max_{Q_1,Q_2} \; P_1(Q_1)Q_1 + P_2(Q_2) - c(Q_1+Q_2) \\
\Longleftrightarrow \; & \max_{Q_1,Q_2} \; (6-Q_1)Q_1 + (6-\tfrac{1}{2}Q_2).
\end{aligned}$$

First order conditions are:

$$\begin{aligned}
\frac{\partial \pi}{\partial Q_1} &= -Q_1 + (6-Q_1) = 0 \\
\frac{\partial \pi}{\partial Q_2} &= -\tfrac{1}{2}Q_2 + (6-\tfrac{1}{2}Q_2) = 0.
\end{aligned}$$

Therefore,

$$Q_1 = 3 \text{ and } P_1 = 3 \quad \text{and} \quad Q_2 = 6 \text{ and } P_2 = 3.$$