Intermediate Microeconomics

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Problem Set 3: suggested solutions

1. Consider an individual whose utility function over wealth is $u(w) = \ln w$.

   (a) What is the individual's Arrow-Pratt measure of absolute risk aversion and the Arrow-Pratt measure of relative risk aversion? How do they change as her wealth changes?

   Solution: Since $u'(w) = \frac{1}{w}$ and $u''(w) = -\frac{1}{w^2}$, we have

$$R_A(w) = -\frac{u''(w)}{u'(w)} = -\frac{-\frac{1}{w^2}}{\frac{1}{w}} = \frac{1}{w} \quad \text{(DARA)}$$

$$R_R(w) = -\frac{wu''(w)}{u'(w)} = -\frac{-w\frac{1}{w^2}}{\frac{1}{w}} = \frac{\frac{1}{w}}{\frac{1}{w}} = 1 \quad \text{(CRRA)}.$$

   (b) Suppose the individual's current wealth is W, and she faces a risk in which she could gain 25% of her wealth with probability $\frac{1}{2}$ or lose 60% of her wealth with probability $\frac{1}{2}$. What is the maximum amount of money that she is willing to pay to avoid this risk?

   Solution: Letting M be the maximum willingness to pay, we have

$$\ln(W - M) \geq \frac{1}{2}\ln(1.25W) + \frac{1}{2}\ln(0.4W) = \ln\left((1.25W)^{\frac{1}{2}}(0.4W)^{\frac{1}{2}}\right) = \ln\left(0.5^{\frac{1}{2}}W\right)$$

$$\implies W - M \geq 0.5^{\frac{1}{2}}W \implies M \leq (1 - \sqrt{.5})W = (1 - 0.707)W = 0.293W.$$

   (c) Suppose the individual did not pay and took the risk and that after the gain or the loss occurs she faces the exact same risk in part (b) again. What is the maximum amount she will pay to avoid the risk now?

   Solution: From part (b), we know that her maximum willingness to pay will be $0.293(1.25W) = 0.366W$ if she had gained and $0.293(0.4W) = 0.117W$ if she had lost.

   (d) Suppose the individual faced the exact same risk in part (b) n times already and is facing it yet again. What is the maximum amount she will pay to avoid the risk now? How about the maximum she is willing to pay as a share of her wealth? Explain this result in relation to part (a).

   Solution: Let m, where $0 \leq m \leq n$, be the number of times the individual gained. Then her current wealth is $W_n = W(1.25)^m(0.4)^{n-m}$. So the maximum amount she is willing to pay is $0.293W_n = 0.293(1.25)^m(0.4)^{n-m}W$.

Note that no matter what her wealth is currently, the maximum she is willing to pay in relative terms (as a share of wealth) is constant 0.293. This is because she has a CRRA utility, which means her attitude toward a risk that are given in relative terms is independent of her wealth level.

2. Consider an individual whose utility function over money is $u(w) = 1 + 2w^{\frac{1}{2}}$.

   (a) Determine the individual's attitude toward risk. Does it depend on w?
   Solution: We have

$$\begin{aligned} u'(w) &= w^{-\frac{1}{2}}, \quad \text{and} \\ u''(w) &= -\tfrac{1}{2}w^{-\frac{3}{2}} < 0 \quad \text{for all } w \text{ (except } w = 0). \end{aligned}$$

   So, the individual is strictly risk-averse at all wealth level.

   (b) Suppose the individual has initial wealth ¥W and faces the possible loss of ¥$\frac{W}{2}$. The probability that the loss will occur is $\frac{1}{2}$. Suppose insurance is available at price p, where p is not necessarily the fair price. Find the optimal amount of insurance the individual should buy. You may assume that the solution is interior.
   Solution: The individual solves

$$\begin{aligned} &\max_x \ \frac{1}{2}u(W - px) + \frac{1}{2}u\left(W - \frac{W}{2} - px + x\right) \\ \Longleftrightarrow &\max_x \ \frac{1}{2}\left(2(W - px)^{\frac{1}{2}}\right) + \frac{1}{2}\left(2\left(\frac{W}{2} + (1-p)x\right)^{\frac{1}{2}}\right). \end{aligned}$$

   Assuming $p < 1$, FOC is given by

$$\frac{1}{2}(W - px)^{-\frac{1}{2}}(-p) + \frac{1}{2}\left(\frac{W}{2} + (1-p)x\right)^{-\frac{1}{2}}(1-p) = 0.$$

$$\begin{aligned} (1-p)^{-2}\left(\frac{W}{2} + (1-p)x\right) &= p^{-2}(W - px) \\ (1-p)^{-2}\left(\frac{W}{2}\right) + (1-p)^{-1}x &= p^{-2}W - p^{-1}x \\ (1-p)^{-1}x + p^{-1}x &= Wp^{-2} - \left(\frac{W}{2}\right)(1-p)^{-2}. \end{aligned}$$

$$\begin{aligned} \Longrightarrow x^* &= \frac{Wp^{-2} - \left(\frac{W}{2}\right)(1-p)^{-2}}{(1-p)^{-1} + p^{-1}} = W\left(\frac{\frac{1-p}{p} - \frac{p}{2(1-p)}}{(1-p) + p}\right) \\ &= W\left(\frac{1-p}{p} - \frac{p}{2(1-p)}\right) = \frac{W(p^2 - 4p + 2)}{2p(1-p)}. \end{aligned}$$

(c) Is there a price at which the individual will not want to buy any insurance? If so, find it. If no, explain.

Solution: Using the quadratic formula (and assuming $p < 1$), we obtain

$$x^* = \frac{W(p^2 - 4p + 2)}{2p(1-p)} \leq 0 \qquad \Longrightarrow \qquad p \geq 2 - \sqrt{2} \approx 0.5858.$$

3. Consider an investor whose utility function over money is $u(w) = w^a$, where $0 < a < 1$. The investor can invest in a riskless asset that returns 1 (gross return per \$1 invested) for sure, or a risky asset that returns (gross) 1.1 with probability $\frac{3}{5}$ and 0.9 with probability $\frac{2}{5}$.

(a) Find the investor's Arrow-Pratt measure of absolute risk aversion. Is the investor risk averse, risk neutral, or risk loving? How does her degree of absolute risk aversion change as her wealth changes?

Solution: We have,

$$u'(w) = aw^{a-1} \quad \text{and} \quad u''(w) = a(a-1)w^{a-2}$$

$$\Longrightarrow R_A(w) = -\frac{u''(w)}{u'(w)} = -\frac{a(a-1)w^{a-2}}{aw^{a-1}} = \frac{(1-a)}{w} > 0 \Longrightarrow \text{ stricIty risk averse.}$$

$$\frac{dR_A(w)}{dw} = -(1-a)w^{-2} < 0 \Longrightarrow \text{ DARA.}$$

(b) Suppose the investor's initial wealth is \$$W$. Find the optimal amount to invest in the risky asset (use x to denote the amount invested in the risky asset).

Solution: Let $g(x)$ be the lottery that results from investing \$$x$ in the risky asset. We have

$$g(x) = \begin{cases} W - x + 1.1x = W + 0.1x & \text{with probability } \frac{3}{5} \\ \\ W - x + 0.9x = W - 0.1x & \text{with probability } \frac{2}{5} \end{cases}$$

So the investor solves:

$$\max_x \; U(g(x)) \iff \max_x \; \frac{3}{5}(W + 0.1x)^a + \frac{2}{5}(W - 0.1x)^a$$

Solving FOC yields,

$$\left(\frac{3a(0.1)}{5}\right)(W + 0.1x)^{a-1} - \left(\frac{2a(0.1)}{5}\right)(W - 0.1x)^{a-1} = 0$$

$$\Longrightarrow 3\,(W + 0.1x)^{a-1} = 2\,(W - 0.1x)^{a-1}$$

$$W + 0.1x = \left(\frac{3}{2}\right)^{\frac{1}{1-a}}(W - 0.1x)$$

$$\left(\left(\frac{3}{2}\right)^{\frac{1}{1-a}} + 1\right)(0.1x) = \left(\left(\frac{3}{2}\right)^{\frac{1}{1-a}} - 1\right)W$$

$$x^* = \frac{\left(\left(\frac{3}{2}\right)^{\frac{1}{1-a}} - 1\right)10W}{\left(\frac{3}{2}\right)^{\frac{1}{1-a}} + 1}$$

(c) How does the amount invested in the risky asset changes as the investor's wealth increases? Give a brief explanation of this result (in light of part (a)).

Solution: It increases since $\frac{dx^*}{dW} > 0$. Since the investor has a DARA utility function, the individual becomes less risk averse as her wealth increases. Therefore she takes on more risk by putting greater amount of her wealth into the risky asset.

4. Let $c(q) = 100 + 10q^3 - 20q^2 + 20q$ be a competitive firm's short-run cost function.

(a) A firm's supply curve is a graph showing how much output the firm produces at each (output) price p. Explain why a firm's short-run supply curve is same as its short-run marginal cost curve above the minimum average variable cost.

Solution: As shown in class, conditioned on producing a positive output, a firm produces output level q, where $p = MC(q)$. Thus, the inverse function of the marginal cost function, $MC^{-1}(p)$ is exactly the function that relates price to the firm's output level (that is, it is the "supply function" of part (b) and the marginal cost function itself is the "inverse supply function." Since we graph the supply curve by placing quantity on the horizontal axis (where independent variable would usually go) and price on the vertical axis (where dependent variable would usually go), the supply curve is the graph of the inverse supply function (the marginal cost function $MC(q)$). Next, since in the short run, the firm only produces a positive output level when the price is above its minimum average variable cost, its supply curve is the marginal cost curve above the minimum average variable cost. Finally, you can verify for yourself that when the Marginal cost curve must be upward sloping if it is above the minimum average variable cost (that is, the second order condition for profit maximization is satisfied).

(b) A supply curve can be expressed as a supply function $q(p)$, which gives output as a function of price, or as an inverse supply function $p(q)$, which gives price as a function of output. Find the firm's inverse supply function.

Solution: As noted, a firm's short-run supply curve is the MC curve above the minimum AVC. To find the minimum AVC, set $MC = AVC$:

$$\begin{aligned} 30q^2 - 40q + 20 &= 10q^2 - 20q + 20 \\ 20q^2 - 20q &= 0 \\ q &= 1 \\ \Rightarrow \quad \min AVC &= 10(1)^2 - 20(1) + 20 \;=\; 10. \end{aligned}$$

Setting $p = MC$ yields the inverse supply function;

$$p \;=\; 30q^2 - 40q + 20, \qquad \text{where } q \geq 1.$$

(c) Suppose $p = 5$. Find the firm's profit.

Solution: Since $p < \min AVC$, the firm will shut down and incur a loss equal to the fixed cost, \$100.

(d) Suppose $p = 30$. Find the firm's profit.

Solution: Since $p > \min AVC$, the firm will produce a positive amount. Its production level can be found from the (inverse) supply function:

$$\begin{aligned}
30 &= 30q^2 - 40q + 20 \\
30q^2 - 40q - 10 &= 0 \\
3q^2 - 4q - 1 &= 0 \\
q &= \frac{4 \pm \sqrt{16 - 4(3)(-1)}}{6} = \frac{4 \pm \sqrt{28}}{6} \\
&= \frac{4 \pm 5.29}{6} = 1.55.
\end{aligned}$$

Firm's profit is

$$\begin{aligned}
pq - c(q) &= 30(1.55) - \big(100 + 10(1.55)^3 - 20(1.55)^2 + 20(1.55)\big) \\
&= 46.5 - (100 + 37.24 - 48.05 + 31) = 46.5 - (100 + 20.19) \\
&= -73.69
\end{aligned}$$

The firm is making a loss but is nevertheless willing to produce since the loss is less than the fixed cost, \$100.

5. Suppose a firm's production function is given by $f(L, K) = L^a K^b$, where $a > 0$ and $b > 0$. As usual, let w, r, and p be the price of labor, capital, and the output good, respectively.

(a) Determine the conditions on a and b under which the firm has constant, decreasing, and increasing returns to scale, respectively.

Solution: We have $f(\lambda L, \lambda K) = (\lambda L)^a (\lambda K)^b = \lambda^{a+b} L^a K^b = \lambda^{a+b} f(L, K)$. So, the firm exhibits DRS if $a+b < 1$, CRS if $a+b = 1$, and IRS if $a+b > 1$.

(b) Find the firm's (long-run) cost function.

Solution: "MRTS = price ratio" and output equation yield,

$$\frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}} = \frac{aL^{a-1}K^b}{bL^a K^{b-1}} = \frac{aK}{bL} = \frac{w}{r} \implies K = \frac{bw}{ar} L \quad \text{and} \quad L^a \left(\frac{bw}{ar} L\right)^b = q.$$

Thus,

$$L^* = \left(\frac{ar}{bw}\right)^{\frac{b}{a+b}} q^{\frac{1}{a+b}} \implies K^* = \left(\frac{bw}{ar}\right)\left(\frac{ar}{bw}\right)^{\frac{b}{a+b}} q^{\frac{1}{a+b}} = \left(\frac{bw}{ar}\right)^{\frac{a}{a+b}} q^{\frac{1}{a+b}}$$

$$\begin{aligned}\implies c(q) &= w\left(\frac{ar}{bw}\right)^{\frac{b}{a+b}} q^{\frac{1}{a+b}} + r\left(\frac{bw}{ar}\right)^{\frac{a}{a+b}} q^{\frac{1}{a+b}} \\ &= \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} w^{\frac{a}{a+b}} r^{\frac{b}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}} w^{\frac{a}{a+b}} r^{\frac{b}{a+b}}\right] q^{\frac{1}{a+b}} \\ &= \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}}\right] w^{\frac{a}{a+b}} r^{\frac{b}{a+b}} q^{\frac{1}{a+b}}.\end{aligned}$$

(c) Suppose a and b are such that the firm has decreasing returns to scale. Is the cost function concave, linear, or convex function of q (output quantity)? Find the firm's (long-run) supply function.

Solution: To make the writing concise, let $\gamma = \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}}\right] w^{\frac{a}{a+b}} r^{\frac{b}{a+b}}$ be the "constant" part of the cost function for the remainder of this question so that $c(q) = \gamma q^{\frac{1}{a+b}}$. If $a + b < 1$, then $\frac{1}{a+b} > 1$, which means $c''(q) = \left(\frac{\gamma}{a+b}\right)\left(\frac{1}{a+b} - 1\right) q^{\frac{1}{a+b}-2} > 0$ (except possibly at $q = 0$). So, $c(q)$ is a strictly convex function of q. Solving $p = MC(q)$ yields

$$p = \left(\frac{\gamma}{a+b}\right) q^{\frac{1}{a+b}-1} = \left(\frac{\gamma}{a+b}\right) q^{\frac{1-(a+b)}{a+b}} \implies q^* = \left(\frac{(a+b)p}{\gamma}\right)^{\frac{a+b}{1-(a+b)}}.$$

Note that the second order condition is satisfied since $c(q)$ is strictly convex.

(d) Suppose a and b are such that the firm has increasing returns to scale. Is the cost function concave, linear, or convex function of q? What is the firm's (long-run) profit maximizing output level?

Solution: By above, $c''(q) < 0$ when $a + b > 1$, so the cost function is linear. Since MC is always decreasing the second order condition cannot be satisfied, and the firm always wants to increase its production. That is, $q^* = \infty$ (or, technically, π-maximizing output level does not exit).

(e) Suppose a and b are such that the firm has constant returns to scale. Is the cost function concave, linear, or convex function of q? Find the firm's (long-run) supply function. What does this say about the shape of firm's (long-run) supply curve?

Solution: By above, $c''(q) = 0$ when $a + b = 1$, so marginal cost is constant and the cost function is linear. Solving $p = MC(q)$ in this case yields $p = \frac{\gamma}{a+b}$. Note that $\frac{\gamma}{a+b}$ is also the minimum MC (since MC is constant). Thus,

$$q^* = \begin{cases} \infty \text{ (π-max soln does not exist)} & \text{if } p > \frac{\gamma}{a+b} \\ \text{any output} \geq 0 & \text{if } p = \frac{\gamma}{a+b} \\ 0 \text{ (shuts down)} & \text{if } p < \frac{\gamma}{a+b}. \end{cases}$$

Thus, the supply curve is flat at $p = \frac{\gamma}{a+b}$.