Intermediate Microeconomics

Fall 2024 - M. Chen, M. Pak, and B. Xu

Problem Set 2: suggested solutions

1. Consider a consumer with income $I = 10$ and utility function:

$$u(x_1, x_2) \;=\; x_1 + 2x_2^{\frac{1}{2}}.$$

(a) Suppose prices are $p_1 = 2$ and $p_2 = 1$. Find the consumer's demand.
Solution: We have

$$MRS \;=\; \frac{1}{x_2^{-\frac{1}{2}}} \;=\; x_2^{\frac{1}{2}} \;=\; \frac{p_1}{p_2} \quad\Rightarrow\quad x_2 \;=\; \frac{p_1^2}{p_2^2} \;=\; \frac{4}{1} \;=\; 4.$$

and

$$p_1x_1 + p_2x_2 = I \quad\Rightarrow\quad x_1 \;=\; \frac{I - p_2x_2}{p_1} \;=\; \frac{I - \frac{p_1^2}{p_2}}{p_1} \;=\; \frac{10-4}{2} \;=\; 3$$

(b) Suppose now that the prices are $p_1' = 4$ and $p_2 = 1$. Find the consumer's demand.
Solution: Now, we have

$$MRS \;=\; \frac{1}{x_2^{-\frac{1}{2}}} \;=\; x_2^{\frac{1}{2}} \;=\; \frac{p_1'}{p_2} \quad\Rightarrow\quad x_2 \;=\; \frac{{p_1'}^2}{p_2^2} \;=\; \frac{16}{1} \;=\; 16.$$

and

$$p_1'x_1 + p_2x_2 = I \quad\Rightarrow\quad x_1 \;=\; \frac{I - p_2x_2}{p_1'} \;=\; \frac{I - \frac{{p_1'}^2}{p_2}}{p_1'} \;=\; \frac{10-16}{2} \;<\; 0$$

But since consumption cannot be negative, this means that $x_1 = 0$ (i.e., there are no interior solution). This forces $x_2 = \frac{I}{p_2} = 10$.

(c) Interpret this result.
Solution At the solution, $x_1 = 0$ and $x_2 = 10$, we have

$$\frac{MU_1}{p_1'} \;=\; \frac{1}{p_1'} \;=\; \frac{1}{4} \;<\; .316 \;\approx\; (10)^{-\frac{1}{2}} \;=\; \frac{x_2^{-\frac{1}{2}}}{p_2} \;=\; \frac{MU_2}{p_2}.$$

So, at this consumption bundle, the consumer would like to reduce consumption of good 1 further in favor of good 2. However, this is not possible since the consumer is already consuming zero units of good 1.

2. Andy's life is composed of two periods. In period 1, he works and earns ¥1,000,000. In period 2, he retires and earns ¥0. Andy's utility function over the consumptions in the two periods is given by

$$u(c_1, c_2) = \ln c_1 + \ln c_2.$$

Andy can lend or borrow freely at the interest rate $r = 0.2$

(a) Find Andy's optimal consumption bundle and the savings level.

Solution: Since $I_2 = 0$, the budget constraint (in future values) is

$$(1+r)c_1 + c_2 \; = \; (1+r)I_1.$$

The optimization problem is:

$$\max_{c_1, c_2} \ln c_1 + \ln c_2 \quad \text{s.t.} \quad (1+r)c_1 + c_2 = (1+r)I_1.$$

Setting MRS = "price ratio" yields:

$$MRS = \frac{\frac{\partial u}{\partial c_1}}{\frac{\partial u}{\partial c_2}} \; = \; \frac{\frac{1}{c_1}}{\frac{1}{c_2}} \; = \; \frac{c_2}{c_1} \; = \; 1+r \quad \Rightarrow \quad c_2 = (1+r)c_1$$

Substituting this into the budget equation yields:

$$(1+r)c_1 + (1+r)c_1 = (1+r)I_1.$$

Therefore,

$$\begin{aligned} c_1^* &= \frac{I_1}{2} = \frac{1,000,000}{2} = 500,000, \\ c_2^* &= \frac{(1+r)I_1}{2} = \frac{(1.2)(1,000,000)}{2} = 600,000, \quad \text{and} \\ S^* &= I_1 - c_1^* = ¥500,000. \end{aligned}$$

(b) Suppose that before Andy makes his consumption and savings choice, the government proposes a new policy that will give every retiree ¥100,000. How will this policy, if enacted, affect Andy's consumption and savings choice?

Solution: Since MRS = "price ratio" equation hasn't changed, we still have $c_2 = (1+r)c_1$. Substituting this into the new budget equation:

$$(1+r)c_1 + c_2 = (1+r)I_1 + I_2$$

yields

$$(1+r)c_1 + (1+r)c_1 = (1+r)I_1 + I_2.$$

Therefore,

$$\begin{aligned} c_1^{**} &= \frac{(1+r)I_1+I_2}{2(1+r)} = \frac{1,300,000}{2.4} = 541,666.67, \\ c_2^{**} &= \frac{(1+r)I_1+I_2}{2} = \frac{1,300,000}{2} = 650,000, \quad \text{and} \\ S^{**} &= I_1 - c_1^{**} = ¥458,333.33. \end{aligned}$$

Since the government will provide some retirement benefit, Andy will save less and devote more of his income to current consumption.

3. Consider a two-period consumption model in which an individual's lifetime utility is $u(c_1, c_2) = \min\left\{c_1, \frac{c_2}{a}\right\}$, where c_t is the consumption in period t and $a > 0$. The individual receives income I_1 in period 1 and I_2 in period 2. The price levels in both periods are one ($p_1 = p_2 = 1$).

   (a) Suppose the individual can borrow or save freely at a bank at a net interest rate $r > 0$. Find the values of a for which the individual will be a saver and not a borrower. Explain this result.

   Solution: For a Leontieff utility, we substitute "corner equation", $c_2 = ac_1$, into the budget constraint:

$$(1+r)c_1 + ac_1 = (1+r)I_1 + I_2 \implies c_1^* = \frac{(1+r)I_1+I_2}{1+r+a}.$$

   The individual will be a saver if

$$I_1 > c_1^* \iff I_1 > \frac{(1+r)I_1+I_2}{1+r+a} \iff (1+r+a)I_1 > (1+r)I_1 + I_2 \iff a > \frac{I_2}{I_1}.$$

   As a increases, the consumption in period 2 becomes more important to the individual, and her incentive to save increases. Note that if her income in period 2 relative to period 1, $\frac{I_2}{I_1}$, increases then the need for saving decreases. Thus, the relative importance of period 2 consumption that is need to turn her into a saver also increases.

For parts (b) and (c) below, assume $a = 1$ and $I_1 > I_2$.

   (b) Now, suppose that there is no bank and that the only way to save in this economy is through a government program that allows individuals to save up to 25% of their period 1 income at a net interest rate $g > r$. When will the cap on the amount that can be saved be binding?

   Solution: Ignoring the cap, the budget constraint is the same as part (a), except r is replaced by g. Thus, the optimal consumption is now

$$c_1^G = \frac{(1+g)I_1+I_2}{1+g+a} = \frac{(1+g)I_1+I_2}{2+g}, \text{ provided that this is } \geq 0.75I_1$$

   The cap is binding if $c_1^G = \frac{(1+g)I_1+I_2}{2+g} < \frac{3I_1}{4}$. That is,

$$4(1+g)I_1 + 4I_2 < 3(2+g)I_1 \iff 4I_2 < 2I_1 - gI_1 \iff g < \frac{2I_1 - 4I_2}{I_1}.$$

(c) When will the situation in part (b) be better than the situation in part (a) for the individual?

Solution: For Leontieff utility, a higher period 1 consumption means higher lifetime utility. We have

$$\frac{d}{dx}\left(\frac{(1+x)I_1+I_2}{2+x}\right) = \frac{I_1(2+x)-((1+x)I_1+I_2)}{(2+x)^2} = \frac{I_1-I_2}{(2+x)^2} > 0.$$

So, if the cap is not binding, the government program is better since $c_1^* < c_1^G$. If the cap is binding, the government program will be better if

$$c_1^* = \frac{(1+r)I_1+I_2}{2+r} < \frac{3I_1}{4} = c_1^g \iff r < \frac{2I_1-4I_2}{I_1} \text{ by replacing } g \text{ with } r \text{ in (b).}$$

4. Let $u(x) = \min\{2x_1, x_2\}$.

(a) Find the Hicksian demand.

Solution: Expenditure minimization problem is

$$\min_{x_1,x_2} \; p_1x_1 + p_2x_2 \quad \text{s.t.} \quad \min\{2x_1, x_2\} = u.$$

The solution occurs where

$$(1)\; 2x_1 = x_2 \quad \text{and} \quad (2)\; \min\{2x_1, x_2\} = u.$$

Thus,

$$x_1^h(p_1,p_2,u) = \frac{u}{2} \quad \text{and} \quad x_2^h(p_1,p_2,u) = u.$$

(b) Expenditure function $e(p_1,p_2,u)$ is the minimum amount of money needed to achieve utility level u when prices are p_1 and p_2. Find the expenditure function this utility function.

Solution:

$$e(p_1,p_2,u) = p_1x_1^h(p_1,p_2,u) + p_2x_2^h(p_1,p_2,u) = \frac{p_1u}{2} + p_2u.$$

5. Consider a leisure-consumption model in which an individual is deciding her daily consumption of leisure and the composite consumption good. Her utility is $u(L,Y) = a\ln L + Y$, where L and Y denote the amount of leisure and the composite good, respectively. The individual has 24 hours of time in total. The hourly wage is w, and the price of the consumption good is p.

(a) Find the individual's (Marshallian) demand for leisure and the consumption good.

Solution: Solving the first order condition for utility maximization yields,

$$MRS = \frac{\frac{\partial u}{\partial L}}{\frac{\partial u}{\partial Y}} = \frac{a}{L} = \frac{w}{p} \implies L^* = \frac{ap}{w} \text{ and } Y^* = \frac{24w - wL^*}{p} = \frac{24w - ap}{p}.$$

(b) For what values of a, if any, will the individual not work? Explain this result.

Solution: The individual will not work if $L^* = \frac{ap}{w} \geq 24 \iff a \geq \frac{24w}{p}$. Note that as a increases, the individual's value for leisure relative to the consumption good increases; therefore, her incentive to work decreases, holding everything else constant. Indeed, if $a \geq \frac{24w}{p}$ then her MRS_{LY} at 24 hours of leisure is $\frac{a}{L} \geq \frac{24w}{24p} = \frac{w}{p}$. If the weak inequality holds with equality then that means 24 hours of leisure (zero hours of work) is optimal. If it holds with strict inequality, then it means that the (relative) market price of leisure is cheaper than her (relative) valuation for leisure. Thus, she in fact wants to buy more leisure, but she cannot because she's already consuming the maximum amount of leisure possible.

(c) Suppose the hourly wage increases from $w = 2$ to $w' = 4$ while the price of the composite good remains fixed at $p = 1$. Find the substitution, the income, and the total effects on leisure.

Solution: For this utility function, the Marshallian demand and the Hicksian demands for leisure is determined solely by the "MRS = price ratio" condition. Therefore, they are the same. Thus, assuming interior solution and letting u^0 denote the utility at the original Marshallian demand, $L(w, I)$ and $Y(w, I)$, we have:

$$
\begin{aligned}
TE_L &= x_L(w', I) - x_L(w, I) = \frac{a}{4} - \frac{a}{2} = -\frac{a}{4} \\
SE_L &= h_L(w', u^0) - x_L(w, I) = \frac{a}{4} - \frac{a}{2} = -\frac{a}{4} \\
\implies IE_L &= TE_L - IE_L = 0.
\end{aligned}
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