Intermediate Microeconomics

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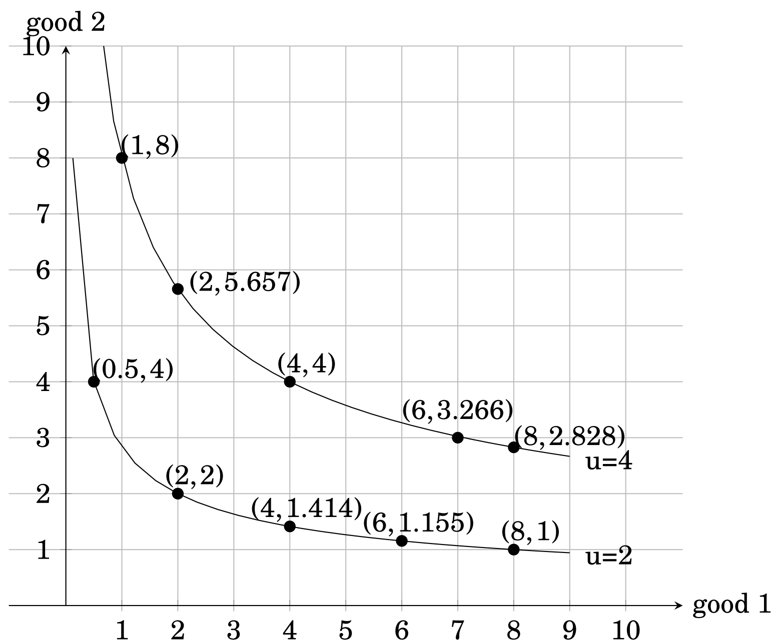
Problem Set 1: suggested solutions

1. Carefully sketch the indifference curves corresponding to the utility functions and the utility levels given below. Be sure to plot several (four or five) points that belong on each indifference curve.

   (a)

$$u(x_1,x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}; \quad u_0 = 2 \text{ and } u_1 = 4.$$

Solution: Solving the identity $x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} = \bar{u}$ for x_2 yields, $x_2 = \left(\frac{\bar{u}}{x_1^{\frac{1}{3}}}\right)^{\frac{3}{2}}$. Substituting utility values $u_0 = 2$ and $u_1 = 4$ for $\bar{u}$ and choosing various values for x_1 yields the following indifference curves.



   (b)

$$u(x_1,x_2) = x_1 x_2^2; \quad u_0 = 8 \text{ and } u_1 = 64.$$

Solution: We obtain the same graph as part (a), except that the lower indifference curve corresponds to utility value 8 and the upper one corresponds to 64.

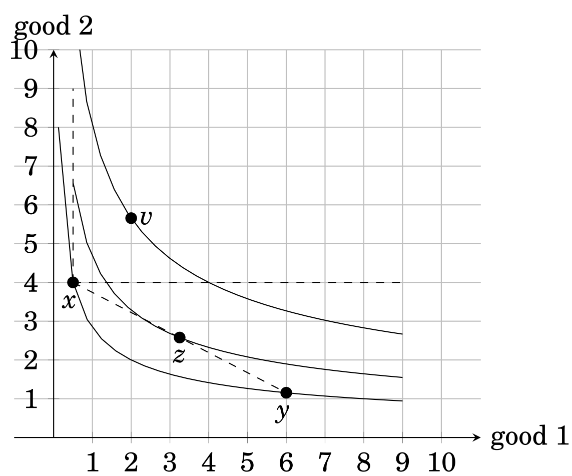
(c)

$$u(x_1,x_2) = \frac{1}{3}\ln x_1 + \frac{2}{3}\ln x_2; \quad u_0 = \ln 2 \text{ and } u_1 = \ln 4.$$

Solution: We obtain the same graph as part (a), except that the lower indifference curve corresponds to utility value $\ln 2$ and the upper one corresponds to $\ln 4$.

(d) Explain how you would convince yourself that these indifference curves belong to the same individual and that the individual's preference ordering is convex and monotone.

Solution: These indifference curves belong to the same individual because they have exactly the same shape, meaning they reflect the same preference ordering. For this class, it is sufficient to argue graphically for monotonicity and convexity. For example, for monotonicity we can pick an arbitrary bundle and show that a bundle that has more everything lies on a higher indifference curve (see points x and v in the figure below). For convexity, we can pick to arbitrary points on a same indifference curve and show that a convex combination of these two bundle lies on a higher (or the same) indifference curve (see points x, y, and z in the figure below).



2. Answer the following.

(a) Suppose $f(\cdot)$ is an increasing function. Show that the utility functions $u(\cdot)$ and $\tilde{u}(\cdot) = f\big(u(\cdot)\big)$ generate the same (Marshallian) demand.

Solution: The utility maximization problem with utility function $u(\cdot)$ is

$$\text{UMP:} \quad \max_{x_1,x_2} u(x_1,x_2) \quad \text{s.t.} \quad p_1x_1 + p_2x_2 \le I$$

while the the utility maximization problem with utility function $\tilde{u}(\cdot)$ is

$$\widetilde{\text{UMP}}: \quad \max_{x_1,x_2} \tilde{u}(x_1,x_2) \quad \text{s.t.} \quad p_1x_1 + p_2x_2 \le I.$$

Because the two problems have exactly the same constraint, all that is needed for the two problems to have the same solution is that $u(\cdot)$ and

$\tilde{u}(\cdot)$ have the same ordering (that is, $u(x) \geq u(y) \iff \tilde{u}(x) \geq \tilde{u}(y)$), which is true since $f(\cdot)$ is an increasing function. To state this more formally, let $B(p_1, p_2, I)$ denote the budget set (which is the same for both problems). Suppose x^* is a solution to UMP. Consider any $x \in B(p_1, p_2, I)$. We have

$$u(x^*) \geq u(x) \implies f(u(x^*)) \geq f(u(x)) \implies \tilde{u}(x^*) \geq \tilde{u}(x).$$

So, x^* is also the solution to $\widetilde{\text{UMP}}$. Conversely, suppose x^* is a solution to $\widetilde{\text{UMP}}$ and consider any $x \in B(p_1, p_2, I)$. We have

$$\tilde{u}(x^*) \geq \tilde{u}(x) \implies f(u(x^*)) \geq f(u(x)) \implies u(x^*) \geq u(x).$$

So, x^* is also the solution to UMP.

(b) Verify the above for the three utility functions in Question 1 by explicitly solving the corresponding utility maximization problems.

Solution:

- For $u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$:

$$MRS = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}}}{\frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}}} = \frac{x_2}{2x_1} = \frac{p_1}{p_2} \implies x_2 = \frac{2p_1}{p_2} x_1.$$

  Substituting this into the budget equation yields:

$$p_1 x_1 + p_2 \left(\frac{2p_1}{p_2} x_1 \right) = I \implies 3p_1 x_1 = I.$$

  So,

$$x_1(p_1, p_2, I) = \frac{I}{3p_1} \quad \text{and} \quad x_2(p_1, p_2, I) = \frac{2p_1}{p_2} x_1(p_1, p_2, I) = \frac{2I}{3p_2}.$$

- For $u(x_1, x_2) = x_1 x_2^2$:

$$MRS = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{x_2^2}{2x_1 x_2} = \frac{x_2}{2x_1} = \frac{p_1}{p_2} \implies x_2 = \frac{2p_1}{p_2} x_1.$$

  Substituting this into the budget equation yields:

$$p_1 x_1 + p_2 \left(\frac{2p_1}{p_2} x_1 \right) = I \implies 3p_1 x_1 = I.$$

  So,

$$x_1(p_1, p_2, I) = \frac{I}{3p_1} \quad \text{and} \quad x_2(p_1, p_2, I) = \frac{2p_1}{p_2} x_1(p_1, p_2, I) = \frac{2I}{3p_2}.$$

- For $u(x_1, x_2) = \frac{1}{3} \ln x_1 + \frac{2}{3} \ln x_2$:

$$MRS = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\frac{1}{3x_1}}{\frac{2}{3x_2}} = \frac{x_2}{2x_1} = \frac{p_1}{p_2} \implies x_2 = \frac{2p_1}{p_2} x_1.$$

Substituting this into the budget equation yields:

$$p_1 x_1 + p_2 \left(\frac{2p_1}{p_2} x_1 \right) = I \implies 3p_1 x_1 = I.$$

So,

$$x_1(p_1, p_2, I) = \frac{I}{3p_1} \quad \text{and} \quad x_2(p_1, p_2, I) = \frac{2p_1}{p_2} x_1(p_1, p_2, I) = \frac{2I}{3p_2}.$$

(c) Notice that the marginal rates of substitution are the same for all three utility functions in part (b). More generally, show that a utility function $u(\cdot)$ and the transformed utility function $\tilde{u}(\cdot) = f(u(\cdot))$, where $f(\cdot)$ is an increasing function, have the same marginal rate of substitution. You may assume both $u(\cdot)$ and $f(\cdot)$ are differentiable.

Solution: We have

$$\text{MRS for } \tilde{u}(\cdot) = \frac{\frac{\partial \tilde{u}}{\partial x_1}}{\frac{\partial \tilde{u}}{\partial x_2}} = \frac{\frac{df}{du}\frac{\partial u}{\partial x_1}}{\frac{df}{du}\frac{\partial u}{\partial x_2}} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \text{MRS for } u(\cdot).$$

3. Consider an individual with utility function

$$u(x_1, x_2) = \min \left\{ \frac{x_1}{a}, \frac{x_2}{b} \right\}.$$

(a) How does the ratio of the optimal consumption of the two goods (specifically, $\frac{x_2(p_1,p_2,I)}{x_1(p_1,p_2,I)}$) change as prices change?

Solution: Recall that for Leontieff utility functions, the optimal consumption occurs at the corner point. Thus,

$$\frac{x_1}{a} = \frac{x_2}{b} \implies \frac{x_2(p_1, p_2, I)}{x_1(p_1, p_2, I)} = \frac{b}{a},$$

which does not depend on prices. That is, the individual always consumes at the same ratio $\frac{b}{a}$ no matter what the prices are.

(b) How does the ratio of optimal consumption of the two goods change as a and b change?

Solution: As seen in part (a), the individual's consumption of good 2 relative to good 1 increases as b increases and decreases a increases.

(c) Suppose the individual has income I and prices are p_1 and p_2 initially. Suppose the price of good 2 increases to $p_2' > p_2$. Let ΔI be the (minimum) additional income that is needed to make the individual equally as well off as before the price change. Without doing any calculation, guess how ΔI changes as b changes. (You really should do this before part (d)).

Solution: Since there is no substitutability between the goods for Leontieff utility, obtaining the same level of utility (at minimum income) must mean that the individual consumes exactly the same bundle before and after the price change. Moreover, as shown in parts (a) and (b), the individual's consumption of good 2 relative to good 1 increases as b increases (that is, higher b moves the optimal consumption bundle up on the budget line). Together, this implies that increase in the price of good 2 will have an higher impact if b is higher. (Try to draw a picture). That is, ΔI should increase with b.

(d) Verify your answer to part (c) with an explicit calculation.

Solution: The individual's initial consumption bundle is found by substituting the corner point equation, $x_2 = \frac{b}{a}x_1$ into the budget equation:

$$p_1 x_1 + p_2\left(\frac{b}{a}x_1\right) = I \implies x_1(p_1,p_2,I) = \frac{aI}{ap_1+bp_2}$$
$$x_2(p_1,p_2,I) = \frac{b}{a}\left(\frac{aI}{ap_1+bp_2}\right) = \frac{bI}{ap_1+bp_2}.$$

Since the individual must consume exactly the same bundle before and after the price change, the new income I' must satisfy

$$\frac{bI}{ap_1+bp_2} = \frac{bI'}{ap_1+bp_2'} \implies I' = \left(\frac{ap_1+bp_2'}{ap_1+bp_2}\right)I \implies \Delta I = \left(\frac{ap_1+bp_2'}{ap_1+bp_2}\right)I - I.$$

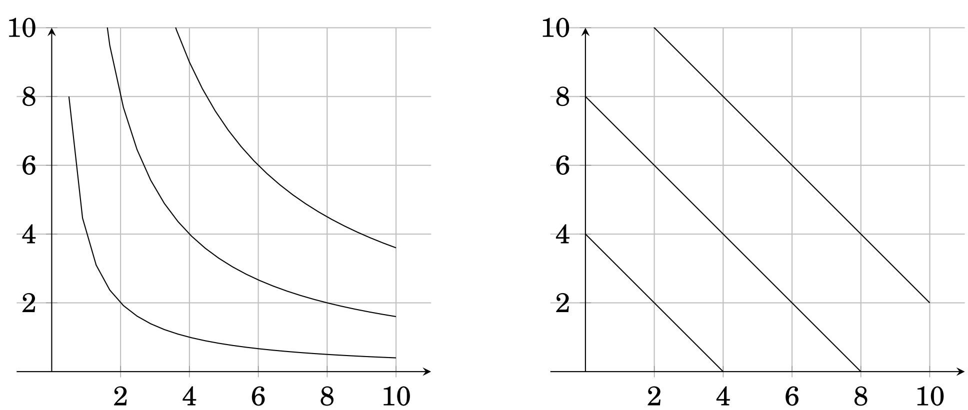
Thus,

$$\frac{\partial \Delta I}{\partial b} = \left(\frac{p_2'(ap_1+bp_2)-(ap_1+bp_2')p_2}{(ap_1+bp_2)^2}\right)I = \left(\frac{ap_1p_2'+bp_2p_2'-ap_1p_2-bp_2p_2'}{(ap_1+bp_2)^2}\right)I$$
$$> 0 \text{ since } p_2' > p_2.$$

4. Consider a firm that has two employees. Employee 1 has a preference ordering that can be represented by utility function $u_1(x_1,x_2) = \ln x_1 + \ln x_2$ while employee 2's preference ordering can be represented by $u_2(x_1,x_2) = x_1 + x_2$.

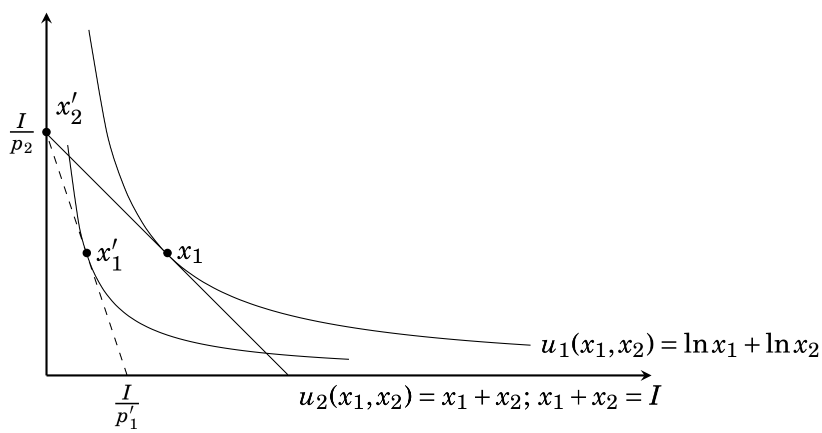
(a) Sketch a few indifference curves (for example, ones going through bundles $(2,2)$, $(4,4)$, and $(6,6)$, respectively) for each employee.

Solution:



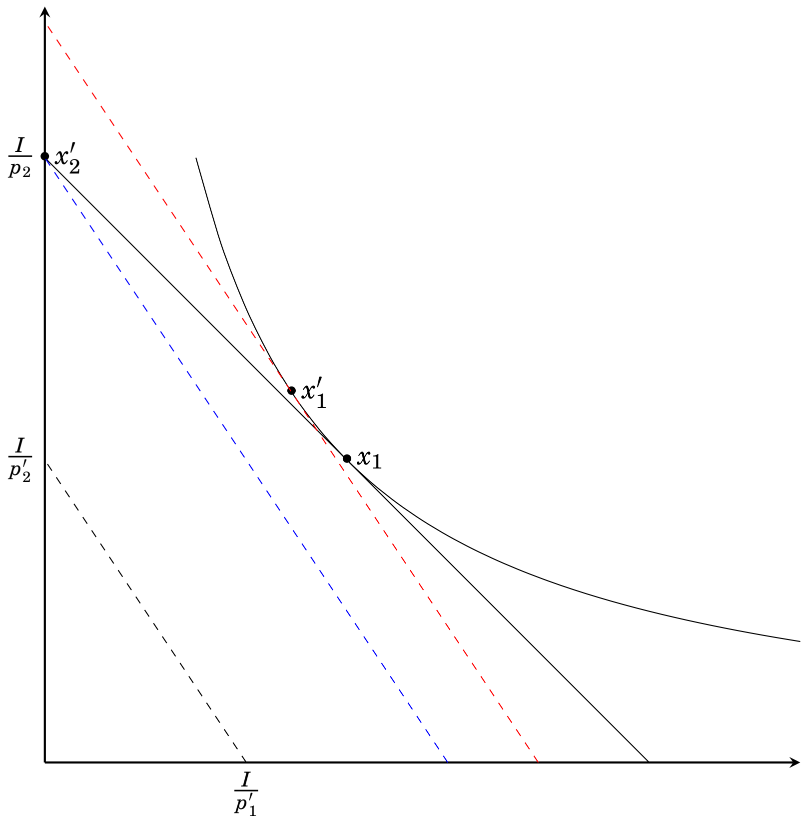
(b) Suppose the firm is located in city C, where the price levels are $(p_1, p_2) = (1,1)$. It needs to send one of the two employees to its branch in city S. However, the price levels in city S are $(p'_1, p'_2) = (3,1)$, so the firm may have to pay an additional salary to ensure that the employee is equally well-off in city S as she was in city C. If the firm wants to minimize the additional salary it needs to pay, which employee should it send? Explain using carefully labeled graphs.

Solution: When prices are $p_1 = p_2 = 1$, employee 2's indifference curve has the same slope as the budget line. Therefore, her optimal consumption bundle is anything on the budget line. In contrast, employee 1 will consume at point x_1 in the figure below. At prices $p'_1 = 3$ and $p'_2 = 1$, the budget line becomes the dashed line. At the dashed budget line, employee 2 can still consume bundle x'_2 which gives her exactly the same utility as before. So there is no need to compensate her. In contrast, individual 1 will have to consume x'_1 which gives him lower utility. Therefore, the firm must compensate him by giving him an additional salary. Thus, to minimize salary, the firm should send employee 2. (Note that figure below assumes that the two employees have the same salary currently (in city C) but the argument does not depend on that).



(c) What if the price levels in city S were $p'' = (3,2)$? Explain using carefully labeled graphs.

solution: (Now the answer depends on the employees' current income. This solution assumes that they have the same salary. However, solutions makings different assumptions are accepted, provided that the reasoning is correct.) When prices are $p'' = (3,2)$, employee 2's budget line has to be the blue dashed line to be equally as well off as before (see the figure below). In contrast, employee 1's budget line has to be the red dashed line (which is tangent to his original indifference curve at bundle x'_1) to be equally as well off as before. Since the blue dashed line is lower than the red dashed line, employee 2 requires less compensation. Thus, the firm should still send employee 2.



5. Consider a consumer whose utility function over electricity (E) and composite good (Y) is given by:

$$u(E,Y) = E^2 Y.$$

   (a) Suppose the price of electricity is \$10, the price of composite good is $p_Y =$ \$5, and income is \$400. Find the consumer's optimal consumption bundle.

   Solution: Since we will need to find the optimal bundle for different prices in this question, we will first find the optimal demand as a function of prices and income (i.e., demand function). Then we will evaluate at the function as given prices and income.

   Setting MRS equal to the price ratio yields

$$MRS = \frac{\frac{\partial u}{\partial E}}{\frac{\partial u}{\partial Y}} = \frac{2EY}{E^2} = \frac{2Y}{E} = \frac{p_E}{p_Y} \implies Y = \frac{p_E}{2p_Y}E.$$

   Substituting this into the budget equation $p_E E + p_Y Y = I$ yields

$$p_E E + p_Y\left(\frac{p_E}{2p_Y}E\right) = I \implies p_E E + \frac{p_E}{2}E = I \implies \frac{3p_E}{2} = I \implies E^* = \frac{2I}{3p_E}.$$

   So,

$$Y^* = \frac{p_E}{2p_Y}E^* = \frac{p_E}{2p_Y}\left(\frac{2I}{3p_E}\right) = \frac{I}{3p_Y}.$$

   Thus, when $p_E = 10$, $p_Y = 5$, and $I = 400$, we have

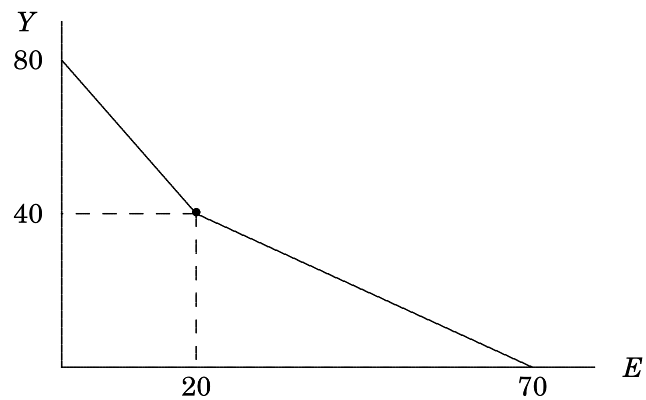
$$E^* = \frac{2(400)}{3(10)} = \frac{800}{30} = 26.67 \quad \text{and} \quad Y^* = \frac{400}{3(5)} = \frac{400}{15} = 26.67.$$

(b) Suppose the government wants to encourage greater use of electricity by subsidizing consumption beyond the first 20kWh. In particular, suppose the price of electricity is \$10 for first 20kWh and \$4 thereafter, while the price of the composite good remains the same at $p_Y = \$5$. Carefully sketch the budget set for the consumer who has income \$400.

Solution: By solving

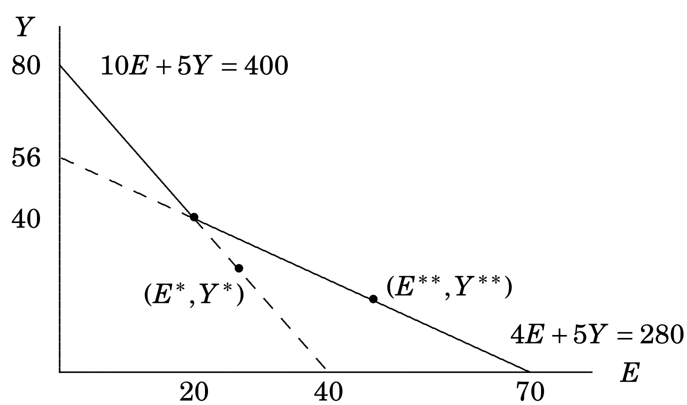
$$10(20) + 5Y = 400 \quad \Longrightarrow \quad Y = 40,$$

we see that the "kink" on the budget line occurs at $x = 20$ and $Y = 40$.



(c) Find the consumer's utility maximizing bundle under the pricing scheme given in part (b).

Solution: One way to approach this problem is to extend the upper and lower portion of the budget line, solve the optimal demand for each of these "budget lines," and check which of the two yields the solution. Extending the budget lines yield:



The upper budget equation $10E + 5Y = 400$ is the same budget equation as part (a). So,

$$E^* = 26.67 \quad \text{and} \quad Y^* = 26.67.$$

Using the lower budget equation $4E + 5Y = 280$ yields

$$E^{**} = \frac{2(280)}{3(4)} = \frac{560}{12} = 46.67 \quad \text{and} \quad Y^{**} = \frac{280}{3(5)} = \frac{280}{15} = 18.67.$$

Looking at the graph above, we see that (E^*, Y^*) is not on the actual kinked line, which is the real budget line for the consumer, unlike (E^{**}, Y^{**}). So, the optimal bundle is $(E^{**}, Y^{**}) = (46.67, 18.67)$.