Intermediate Microeconomics

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Midterm examination: suggested solutions

1. [20] An individual has utility function $u(x_1, x_2) = x_1 + 2x_2^{\frac{1}{2}}$ and income I.

   (a) [5] Find the individual's Marshallian demand, assuming that the solution will be interior (that is, $x_1 > 0$ and $x_2 > 0$).
   Solution:

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{1}{x_2^{-\frac{1}{2}}} = x_2^{\frac{1}{2}} = \frac{p_1}{p_2} \Longrightarrow x_2^* = \left(\frac{p_1}{p_2}\right)^2 \Longrightarrow x_1^* = \frac{I - p_2\left(\frac{p_1}{p_2}\right)^2}{p_1} = \frac{I}{p_1} - \frac{p_1}{p_2}.$$

   (b) [5] Find the individual's Hicksian demand, assuming that the solution will be interior.
   Solution:

$$x_2^h = \left(\frac{p_1}{p_2}\right)^2 \Longrightarrow x_1 + 2\left(\left(\frac{p_1}{p_2}\right)^2\right)^{\frac{1}{2}} = u \Longrightarrow x_1^h = u - \frac{2p_1}{p_2}.$$

   (c) [6] Let $I = 10$. Suppose the price of good 2 rises from $p_2 = 1$ to $p_2' = 2$ while the price of good 1 remains at $p_1 = 2$. Find the substitution effect, the income effect, and the total effect on the two goods.
   Solution: We have $x_1^o = \frac{10}{2} - \frac{2}{1} = 3$, $x_2^o = \left(\frac{2}{1}\right)^2 = 4$, and $u^o = 3 + 2\sqrt{4} = 7$. In addition, $x_1^n = \frac{10}{2} - \frac{2}{2} = 4$, $x_2^n = \left(\frac{2}{2}\right)^2 = 1$. Thus,

$$x_1^h(p_1, p_2', u^o) = 7 - 2\left(\frac{2}{2}\right) = 5 \Longrightarrow x_2^h(p_1, p_2', u^o) = \left(\frac{2}{2}\right)^2 = 1$$
$$\Longrightarrow SE_1 = x_1^h - x_1^o = 5 - 3 = 2 \quad \text{and} \quad SE_2 = x_2^h - x_2^o = 1 - 4 = -3.$$
$$IE_1 = x_1^n - x_1^h = 4 - 5 = -1 \quad \text{and} \quad IE_2 = x_2^n - x_2^h = 1 - 1 = 0.$$
$$TE_1 = x_1^n - x_1^o = 4 - 3 = 1 \quad \text{and} \quad TE_2 = x_2^n - x_2^o = 1 - 4 = -3.$$

   (d) [4] Suppose instead the prices are such that the individual does not consume any good 1. What is the relationship between the marginal rate of substitution and the prices of the goods in this case? Given an interpretation of the relationship.
   Solution: We need $MRS \leq \frac{p_1}{p_2}$ at $x_1 = 0$ (which means $\frac{I}{p_1} \leq \frac{p_1}{p_2}$) so that the individual (weakly) wants to reduce consumption of good 1 further but is unable to do so because she is already consuming zero amount of good 1.

2. [20] An individual lives for two periods and has utility function $u(c_1,c_2) = c_1^a c_2^b$. The individual earns I_1 in period 1 and I_2 in period 2. The price level in each period is one ($p_1 = p_2 = 1$), and the individual can save or borrow freely at net interest rate $r > 0$.

   (a) [10] Determine the minimum interest rate r^* at which the individual will be a saver.

   Solution: The individual's utility maximization problem is

$$\max_{c_1,c_1} c_1^a c_2^b \quad \text{s.t.} \quad (1+r)c_1 + c_2 = (1+r)I_1 + I_2.$$

   Using the Marshallian demand formula for C-D utility function yields,

$$c_1 = \frac{a}{a+b}\left(\frac{(1+r)I_1 + I_2}{1+r}\right) \qquad \text{and } c_2 = \frac{b}{a+b}\big((1+r)I_1 + I_2\big)$$

   Individual will save if and only if $c_2 > I_2$ (or equivalently, $c_1 < I_1$).

$$\frac{b}{a+b}\big((1+r)I_1 + I_2\big) > I_2 \iff b(1+r)I_1 > (a+b)I_2 - bI_2 \iff (1+r) > \frac{aI_2}{bI_1}.$$

   (b) [5] Suppose the individual is currently a saver and interest rate rises. Will the individual be better off? What if the individual was a borrower? You may answer mathematically, graphically, or verbally.

   Solution: If an individual is a saver, she is unambiguously better off if interest rate goes up since her period 2 income will go up even if she does not change her savings. That is, it is now possible her to increase c_2 without lowering c_1. Whether a borrower is better off or not depends on whether the new inter-temporal budget set intersects the individual's pre-interest-rate-increase indifference curve. She is worse off if interest rate goes up slightly because her debt becomes costlier to service. That is, the indifference curve corresponding to her original consumption is outside her new budget set. However, if the interest rate goes up sufficiently, budget set will intersect her original indifference curve. In that case, she will become a saver and her lifetime utility will increase.

   (c) [5] Let $a = b = 1$, $I_1 = 10$, and $I_2 = 0$. Suppose interest rate falls from r to r'. What is the amount of additional income the individual needs in period 1 to be equally well off as before?

   Solution: In this case, we have $c_1 = \frac{I_1}{2}$, $c_2 = \frac{(1+r)I_1}{2}$, which means $u(c_1,c_2) = \frac{(1+r)I_1^2}{4}$. Thus, we need

$$u^o = \frac{(1+r)100}{4} = \frac{(1+r')(I_1')^2}{4} = u^n \implies I' = 10\sqrt{\frac{1+r}{1+r'}} \implies I_1' - I_1 = 10\left(\sqrt{\frac{1+r}{1+r'}} - 1\right).$$

3. [20] Consider an insurance problem where a strictly risk averse individual with initial wealth W faces a possible loss $L < \frac{W}{2}$. The probability of loss is α, where $0 < \alpha < \frac{1}{2}$. Suppose insurance is available at price p RMB per unit, where 1 unit of insurance pays the insuree 0.5 RMB if the loss occurs and nothing otherwise.

   (a) [4] Determine the "actuarily" fair price for the insurance. What is the amount of insurance that is necessary to fully insure against loss?

   Solution:

$$E[\text{profit}] = px - (\alpha(0.5x) + (1-\alpha)(0x)) = 0 \implies p = 0.5\alpha.$$

   Since each unit only pays 0.5 RMB when the loss occurs, $x = 2L$ is the full insurance.

   (b) [12] Suppose $p = \alpha$ so that the price of insurance is equal to the probability of loss. Will the individual under insure, fully insure, or over insure in this case?

   Solution: We have

$$\begin{aligned} U(x) &= \alpha u\big(W - L + (0.5 - p)x\big) + (1-\alpha)u\big(W - px\big) \\ &= \alpha u\big(W - L + (0.5 - \alpha)x\big) + (1-\alpha)u\big(W - \alpha x\big) \\ U'(x) &= \alpha u'\big(W - L + (0.5 - \alpha)x\big)(0.5 - \alpha) - (1-\alpha)u'\big(W - \alpha x\big)\alpha. \end{aligned}$$

   FOC is (SOC is satisfied since $u'' < 0 \implies U'' < 0$):

$$\begin{aligned} u'\big(W - L + (0.5 - \alpha)x\big) &= \frac{1-\alpha}{0.5-\alpha} u'\big(W - \alpha x\big) \qquad (*) \\ \implies u'\big(W - L + (0.5 - \alpha)x\big) &> u'\big(W - \alpha x\big) \\ \implies W - L + (0.5 - \alpha)x &< W - \alpha x \quad \text{since } U' \text{ is decreasing} \\ \implies x^* &< \frac{L}{0.5} = 2L. \end{aligned}$$

   Thus, the individual will under insure.

   (c) [4] Show whether it's possible that the individual does not buy any insurance.

   Solution: Buying no insurance ($x^* = 0$) would be a solution if $U'(0) \le 0$. Substitution $x = 0$ into $(*)$ above yields,

$$u'\big(W - L\big) \le \frac{1-\alpha}{0.5-\alpha} u'(W) \iff u'(W) \ge \underbrace{\frac{0.5-\alpha}{1-\alpha}}_{<1} u'\big(W - L\big).$$

   Thus, if marginal utility of wealth does not decline too rapidly, meaning the individual is not too risk averse, the individual will buy no insurance.

4. [20] Bob's utility of wealth is $u(w) = w^{\frac{1}{2}}$, and his initial wealth is 100 RMB. Let L be a lottery that pays 60 RMB with probability $\frac{1}{2}$ and 0 RMB with probability $\frac{1}{2}$. Let p be the price of the lottery.

(a) [5] Stuart's utility of wealth is $u(w) = w^a$, where $a > 0$. Find the value of a for which Bob will be more risk averse than Stuart.

Solution: Bob will be more risk averse if his degree of absolute risk aversion is higher than Stuart's

$$R_A^B = -\frac{-\frac{1}{4}w^{-\frac{3}{2}}}{\frac{1}{2}w^{-\frac{1}{2}}} = \frac{1}{2w} \quad \text{and} \quad R_A^S = -\frac{a(a-1)w^{a-2}}{aw^{a-1}} = \frac{(1-a)}{w}$$

$$\implies R_A^B > R_A^S \iff \frac{1}{2} > 1-a \iff a > \frac{1}{2}.$$

(b) [5] Find the maximum price at which Bob will buy the lottery.

Solution: Bob will buy the lottery if and only if

$$U(\text{buy}) = \frac{1}{2}u(100-p+60) + \frac{1}{2}u(100-p) \geq u(100) = U(\text{not buy})$$

$$\iff (160-p)^{\frac{1}{2}} + (100-p)^{\frac{1}{2}} \geq 2\left(100^{\frac{1}{2}}\right) \iff (160-p)^{\frac{1}{2}} > 20-(100-p)^{\frac{1}{2}}$$

$$160-p \geq 400 - 40(100-p)^{\frac{1}{2}} + (100-p) \iff 40(100-p)^{\frac{1}{2}} \geq 340$$

$$\implies 100-p \geq \left(\frac{340}{40}\right)^2 \implies p \leq 100 - 8.5^2 = 27.75.$$

(c) [6] Suppose Bob buys the lottery and wins 60 RMB in the lottery. Suppose he is then offered the same lottery again. What is the maximum price at which Bob will buy the lottery now? Compare the result with part (b) and interpret.

Solution: Since Bob has decreasing absolute risk aversion, the maximum price he is willing to pay should decreasing in in his wealth, as seen below.

$$U(\text{buy}) = \frac{1}{2}u(W-p+60) + \frac{1}{2}u(W-p) \geq u(W) = U(\text{not buy})$$

$$\iff (W+60-p)^{\frac{1}{2}} + (W-p)^{\frac{1}{2}} \geq 2\left(W^{\frac{1}{2}}\right)$$

$$\iff W+60-p \geq 4W + (W-p) - 4(W^{\frac{1}{2}})(W-p)^{\frac{1}{2}} \iff 4(W^{\frac{1}{2}})(W-p)^{\frac{1}{2}} \geq 4W-60$$

$$\iff (W-p)^{\frac{1}{2}} \geq W^{\frac{1}{2}} - \frac{15}{W^{\frac{1}{2}}} \iff W-p \geq W + \frac{225}{W} - 30 \iff p \leq 30 - \frac{225}{W}.$$

In particular, the maximum Bob is willing to pay is now $30 - \frac{225}{160} = 30 - 1.406 = 28.594$ RMB.

(d) [4] Suppose Kevin has the same utility function as Bob, but he is wealthier than Bob. Is there a price at which Bob will buy the lottery but Kevin will not?

Solution: Since Bob and Kevin have the same DARA utility function and Kevin is wealthier, the maximum Kevin is willing to pay, p_K, is always higher than Bob's, p_B. This means $p < p_B \implies p < p_K$, so there is no such a price.

5. [20] A firm's production function is $f(L,K) = \min\left\{\frac{L^{\frac{1}{2}}}{a}, \frac{K^{\frac{1}{2}}}{b}\right\}$, where $a > 0$.

(a) [4] Determine the firm's returns to scale.

Solution: The firm has DRS because

$$f(\lambda L, \lambda K) = \min\left\{\frac{(\lambda L)^{\frac{1}{2}}}{a}, \frac{(\lambda K)^{\frac{1}{2}}}{b}\right\} = \lambda^{\frac{1}{2}} f(L,K) < f(L,K) \text{ when } \lambda > 1.$$

(b) [6] Suppose in the short run, capital is fixed at $\bar{K} = 100$. Assuming that the firm wants to produce less than $\frac{10}{b}$, find the firm's marginal product of labor, average product of labor, and short-run cost function.

Solution: When capital is fixed at $\bar{K} = 100$, we have

$$f(L,\bar{K}) = \min\left\{\frac{L^{\frac{1}{2}}}{a}, \frac{10}{b}\right\} = \frac{L^{\frac{1}{2}}}{a} \implies MP_L = \frac{1}{2aL^{\frac{1}{2}}} \quad \text{and} \quad AP_L = \frac{1}{aL^{\frac{1}{2}}}.$$

$$\frac{L^{\frac{1}{2}}}{a} = q \implies L^* = a^2q^2 \implies c(w,r,q) = wa^2q^2 + 100r.$$

(c) [5] Find the firm's long-run cost function.

Solution: For Leontieff production functions, we use the "corner point equation" and the output equation ($\frac{L^{\frac{1}{2}}}{a} = q$) to find the long-run input demands.

$$\frac{L^{\frac{1}{2}}}{a} = \frac{K^{\frac{1}{2}}}{b} \implies K = \left(\frac{b}{a}\right)^2 L \implies L^* = a^2q^2 \text{ and } K^* = b^2q^2$$
$$\implies c(w,r,q) = (wa^2 + rb^2)q^2.$$

(d) [5] Find the firm's long-run profit maximizing output level, and describe the shape of the firm's supply curve.

Solution: Since $c(w,r,q)$ is convex in q ($c'' > 0$), the second order conditioned is satisfied. Thus, we need to apply only the first order condition:

$$p = MC = 2(wa^2 + rb^2)q \implies q^* = \frac{p}{2(wa^2 + rb^2)}.$$

The supply curve (more precisely, the inverse supply curve) is an upward-sloping straight line with slope $2(wa^2 + rb^2)$.