Intermediate Microeconomics

Exercises: Game Theory

Fall 2024 - M. Chen, M. Pak, and B. Xu

1. Consider the following 2 player game:

		player 2		
		L	C	R
player 1	A	10,5	10,1	4,2
	B	7,9	5,8	2,7
	C	5,1	5,0	5,5
	D	3,10	7,9	0,10

(a) Find the strictly dominant strategies solution, if any.

Solution: P2 does not have a dominant strategy, so there is no DSS.

(b) Find the iterated elimination of strictly dominated strategies solution, if any.

Solution: After eliminating C, B, D in this order, no further elimination is possible. So, there is no IEsDSS.

(c) Find all the pure strategy Nash equilibria, if any.

Solution: (A,L) and (C,R).

(d) Find all other (i.e., mixed) Nash equilibria, if any.

Solution: Eliminating strictly dominated strategies, C for P2 and B and D for player 2 leaves following reduced game:

		player 2	
		L	R
player 1	A	10,5	4,2
	C	5,1	5,5

To find P1's strategy, set P2's payoffs equal:

$$\begin{aligned} \pi_2((p,1-p),L) &= \pi_2((p,1-p),R) \\ \Rightarrow 5p+1(1-p) &= 2p+5(1-p) \\ \Rightarrow 4p+1 &= 5-3p \\ \Rightarrow p &= \frac{4}{7}. \end{aligned}$$

To find P2's strategy, set P1's payoff equal:

$$\begin{aligned}\pi_1(A,(q,1-q)) &= \pi_2(C,(1,1-q))\\ \Rightarrow 10q+4(1-q) &= 5q+5(1-q)\\ \Rightarrow 10q+4-4q &= 5q+5-5q\\ \Rightarrow q &= \frac{1}{6}.\end{aligned}$$

So, the Nash equilibrium of the original game is:

$$\left(\left(\frac{4}{7},0,\frac{3}{7},0\right),\left(\frac{1}{6},0,\frac{5}{6}\right)\right).$$

2. Consider the following two player game:

player 1 \ player 2	t_1	t_2	t_3	t_4
s_1	10,8	7,2	10,9	5,10
s_2	3,2	2,4	2,5	7,6
s_3	1,7	7,10	4,9	3,6
s_4	10,5	8,7	9,8	4,8

(a) Find the strictly dominant strategies solution, if any.

Solution: None.

(b) Find the iterated elimination of strictly dominated strategies solution, if any.

Solution: Eliminating t_1, s_3, t_2, s_4, t_3, and s_1 in this order yields (s_2,t_4) as the IEsDSS.

(c) Find all the Nash equilibria, if any.

Solution: (s_2,t_4) is also the unique Nash Equilibrium.

3. Consider the following two player game:

player 1 \ player 2	L	R
A	0,1	4,10
D	5,0	1,−1

(a) Find all the pure strategy Nash equilibria, if any.

Solution: There are two pure strategy Nash equilibria (A,R) and (D,L).

(b) Find all the mixed strategy Nash equilibria, if any.

Solution: We set

$$\begin{aligned}
\pi_1(A,(q,1-q)) &= \pi_1(D,(q,1-q)) \\
4(1-q) &= 5q+(1-q) \\
4-4q &= 4q+1 \\
q &= \tfrac{3}{8}. \\
\text{and} \quad \pi_2((p,1-p),L) &= \pi_1((p,1-p),R) \\
p &= 10p-(1-p) \\
p &= 11p-1 \\
p &= \tfrac{1}{10}.
\end{aligned}$$

So, the mixed strategy Nash equilibrium is $\left(\left(\frac{1}{10},\frac{9}{10}\right),\left(\frac{3}{8},\frac{5}{8}\right)\right)$.

4. Consider the following two player game:

		player 2 L	player 2 R
player 1	T	5,2	0,1
player 1	B	3,0	3,3

(a) Find all the pure strategy Nash equilibria, if any.

Solution: There are two pure strategy Nash equilibria (T,L) and (B,R).

(b) Find all the mixed strategy Nash equilibria, if any.

Solution: We set

$$\begin{aligned}
\pi_1(T,(q,1-q)) &= \pi_1(B,(q,1-q)) \\
5q &= 3q+3(1-q) \\
q &= \tfrac{3}{5}. \\
\text{and} \quad \pi_2((p,1-p),L) &= \pi_1((p,1-p),R) \\
2p &= p+3(1-p) \\
2p &= 3-2p \\
p &= \tfrac{3}{4}.
\end{aligned}$$

So, the mixed strategy Nash equilibrium is $\left(\left(\frac{3}{4},\frac{1}{4}\right),\left(\frac{3}{5},\frac{2}{5}\right)\right)$, and the equilibrium payoff is 3 for player 1 and $\frac{6}{4}$ for player 2.

5. Suppose there are N individuals and one park in the universe. Each individual likes spending time in the park. Letting x_i denote the time individual i spends in the park, assume

$$u_i(x_1,\ldots,x_N) = x_i\left(100-\sum_{j=1}^{N} x_j\right).$$

Note that, in particular, each individual's enjoyment is a decreasing function of the presence of others in the park (say, due to crowding effects).

(a) Find the Nash equilibrium of this game. You may assume that equilibrium $(x_1^*, \ldots, x_n^*)$ is symmetric. I.e., $x_1^* = \ldots = x_N^*$.
Solution:

$$\max_{x_i} \; x_i(100 - x_i - \sum_{j \neq i}^{N} x_j)$$

$$\text{FOC: } 100 - 2x_i - \sum_{j \neq i}^{N} x_j = 0$$

Since equilibrium is symmetric, ie $x_i^* = x_j^*$, so

$$x_i^* = \frac{100 - (N-1)x_i^*}{2}$$

$$x_i^* = \frac{100}{N+1}$$

(b) Let $N = 2$. By socially optimal outcome $(\hat{x}_1, \hat{x}_2)$, we mean the values of x_1 and x_2 that maximizes the sum of the individuals' payoffs. Find the socially optimal outcome. You may assume that the solution is symmetric.
Solution:

$$\begin{aligned} u_1(x_1, x_2) &= x_1(100 - (x_1 + x_2)) \\ u_2(x_1, x_2) &= x_2(100 - (x_1 + x_2)) \end{aligned}$$

Social planner's problem is given by:

$$\begin{aligned} & \max_{x_1, x_2} \; u_1(x_1, x_2) + u_2(x_1, x_2) \\ \Leftrightarrow \quad & \max_{x_1, x_2} \; 100(x_1 + x_2) - 2x_1x_2 - x_1^2 - x_2^2 \\ \text{FOC:} \quad & 100 - 2x_2 - 2x_1 = 0 \end{aligned}$$

Since solution is symmetric, ie, $x_1^* = x_2^*$, we have

$$x_1^* = x_2^* = 25$$

(c) Compare this with the Nash equilibrium outcome when $N = 2$ and give an intuitive explanation of your result.
Solution: From part (a): when $N = 2, x_i^* = \frac{100}{3} = 33.33$. This gives $u_1^* = u_2^* \simeq 1122$ in part (a), where as $u_1^* = u_2^* = 1250$ in part (b). So, intuitively this means that everyone maximizing their own utility without regard for others leads to each one staying in the park longer than socially optimal level. That is, if everyone would spend a little less time in the park, all of them would enjoy a larger utility.

6. Consider the following version of Hotelling competition. There are two vendors selling an identical good at identical price. Since the vendors are identical, the customers, each of whom wants only one unit of the good, will buy from the vendor that is closer to them. To keep things simple, assume that customers are located uniformly along an interval $[0,1]$. That is, each point $x \in [0,1]$ represents one customer's location. When a customer is indifferent between the two vendors, she choses one with equal probability.

   The good costs nothing to the vendor, so the vendors care only about getting as large share of the customers as possible and compete by choosing the location of their businesses. As an example, if vendor 1 chooses location $s_1 \in [0,1]$ and vendor 2 chooses location $s_2 \in [0,1]$, where $s_1 < s_2$, then vendor 1 gets all the customers between 0 and $\frac{s_1+s_2}{2}$ while vendor 2 gets all the customers between $\frac{s_1+s_2}{2}$ and 1. The corresponding payoff are $\frac{s_1+s_2}{2}$ for vendor 1 and $1-\frac{s_1+s_2}{2}$ for vendor 2.

   To summarize, the strategy sets of this game are

$$S_1 = [0,1] \quad \text{and} \quad S_2 = [0,1]$$

   and the payoff functions are

$$\pi_1(s_1,s_2) = \begin{cases} \frac{s_1+s_2}{2} & \text{if } s_1 < s_2 \\ \frac{1}{2} & \text{if } s_1 = s_2 \\ 1-\frac{s_1+s_2}{2} & \text{if } s_1 > s_2 \end{cases}$$

$$\pi_2(s_1,s_2) = \begin{cases} 1-\frac{s_1+s_2}{2} & \text{if } s_1 < s_2 \\ \frac{1}{2} & \text{if } s_1 = s_2 \\ \frac{s_1+s_2}{2} & \text{if } s_1 > s_2 \end{cases}$$

   (a) Show that strategy profile $(s_1^*, s_2^*) = (\frac{1}{2}, \frac{1}{2})$ is a Nash equilibrium.

   Solution: For any $s_1 < s_1^*$,

$$\pi_1(s_1, s_2^*) = \frac{s_1 + s_2^*}{2} = \frac{s_1 + \frac{1}{2}}{2} < \frac{1}{2} = \pi_1(s_1^*, s_2^*).$$

   For any $s_1 > s_1^*$,

$$\pi_1(s_1, s_2^*) = 1 - \frac{s_1 + s_2^*}{2} = 1 - \frac{s_1 + \frac{1}{2}}{2} < \frac{1}{2} = \pi_1(s_1^*, s_2^*).$$

   Therefore, $s_1^* \in BR_1(s_2^*)$. By similar reasoning, $s_2^* \in BR_2(s_1^*)$. So, (s_1^*, s_2^*) is a Nash equilibrium.

   (b) Are there any other Nash equilibria?

   Solution: First, note that there's no Nash equilibrium $(\hat{s}_1, \hat{s}_2)$ in which $\hat{S}_1 = \hat{S}_2 < \frac{1}{2}$. To see this, note that for any $s_1 \in (\hat{S}_2, \frac{1}{2})$,

$$\pi_1(s_1, \hat{s}_2) > \frac{1}{2} = \pi_1(\hat{s}_1, \hat{s}_2).$$

Likewise, there is no Nash equilibrium $(\hat{s}_1, \hat{s}_2)$ in which $\hat{S}_1 = \hat{S}_2 > \frac{1}{2}$ since for any $s_1 \in (\frac{1}{2}, S_2)$,

$$\pi_1(s_1, \hat{s}_2) > \frac{1}{2} = \pi_1(\hat{s}_1, \hat{s}_2).$$

Next, there's no Nash equilibrium $(\hat{s}_1, \hat{s}_2)$ in which $\hat{S}_1 < \hat{S}_2$. To see this, let $S_1 = \frac{\hat{S}_1 + \hat{S}_2}{2}$. Then,

$$\pi_1(s_1, \hat{s}_2) > \pi_1(\hat{s}_1, \hat{s}_2).$$

(You should verify this with calculation.)

Likewise, for similar reason, there's no Nash equilibrium $(\hat{s}_1, \hat{s}_2)$ in which $\hat{S}_1 > \hat{S}_2$.