# Intermediate Microeconomics

Exercises: Choice under Uncertainty

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## 2 Choice Under Uncertainty

1. Consider a risk-neutral utility function $u(x) = x$.

   (a) Give an example of a strictly increasing function $f(\cdot)$ that will make $f(u(x))$ a strictly concave function.
   **Solution:** Let $f(x) = x^{\frac{1}{2}}$. Then, $f(u(x)) = f(x) = x^{\frac{1}{2}}$, which is strictly concave.

   (b) Give an example of a strictly increasing function $f(\cdot)$ that will make $f(u(x))$ a strictly convex function.
   **Solution:** Let $f(x) = x^2$. Then, $f(u(x)) = f(x) = x^2$, which is strictly concave.

   (c) Why should we care about these results?
   **Solution:** Recall that in consumer choice theory without uncertainty, utility functions were ordinal. That is, transforming a utility function by composing it with a strictly increasing function preserved the preference ordering of goods. Therefore, such transformations had no effect on the consumer's demand function. Once uncertainty is introduced, utility functions are no longer ordinal. Although strictly increasing transformations preserve the ordering of money (more is still better), such transformations can change the consumer's attitude towards risk and, hence, affect their observed behavior. So, we should be more skeptical about our predictions of consumer behavior since it requires more precise formulation of the consumer's utility function.

2. Consider an individual whose utility function over money is $u(w) = w^{\frac{1}{2}}$. The individual is facing a risk of losing ¥100 with probability $\frac{1}{2}$ and nothing with probability $\frac{1}{2}$.

   (a) Suppose the individual's current wealth is ¥1,000. What is the maximum amount of money that she is willing to pay to avoid this risk?

**Solution:** The maximum amount of money she is willing to pay, $M$, must solve

$$\begin{aligned} u(1000-M) &= \frac{1}{2}u(1000-100)+\frac{1}{2}u(1000-0) \\ \sqrt{1000-M} &= \frac{1}{2}\sqrt{900}+\frac{1}{2}\sqrt{1000} \approx 30.81 \\ M &= 1000-949.34 = 50.66 \end{aligned}$$

(b) Suppose the individual's current wealth is ¥10,000. What is the maximum amount of money that she is willing to pay to avoid this risk?

**Solution:** The maximum amount of money she is now willing to pay, $M$, must solve

$$\begin{aligned} u(10000-M) &= \frac{1}{2}u(10000-100)+\frac{1}{2}u(10000-0) \\ \sqrt{10000-M} &= \frac{1}{2}\sqrt{9900}+\frac{1}{2}\sqrt{10000} \approx 99.75 \\ M &= 10000-9949.94 = 50.06 \end{aligned}$$

(c) Now, suppose the individual's current wealth is again ¥1000, but now the individual is facing the prospect of gaining ¥100 with probability $\frac{1}{2}$ and nothing with probability $\frac{1}{2}$. What is the maximum amount of money that she is willing to pay to take on this risk?

**Solution:** The maximum amount of money she is willing to pay, $M$, must solve

$$\begin{aligned} u(1000) = \sqrt{1000} = \frac{1}{2}\sqrt{1100-M}+\frac{1}{2}\sqrt{1000-M} &= \frac{1}{2}u(1100-M)+\frac{1}{2}u(1000-M) \\ \sqrt{1100-M} &= 2\sqrt{1000}-\sqrt{1000-M} \\ 1100-M &= 4000-2(2)\sqrt{1000}\sqrt{1000-M}+(1000-M) \\ \sqrt{1000-M} &= \frac{3900}{126.491} = 30.832 \implies M = 1000-30.832^2 = 49.39. \end{aligned}$$

That is, the decision maker will be willing to pay up to \$49.39 to take on this risk.

(d) Interpret these results.

**Solution:** Parts (a) and (b) shows that an individual's evaluation of uncertainty depends on the initial wealth the individual has. That the absolute value of the amount that needs to be paid in Parts (a) and (c) is different shows that an individual's evaluation of uncertainty involving loss may be different from that involving gain.

3. Suppose, as usual, Elmos utility function over gambles satisfies the expected utility property. Consider two gambles $g$ and $h$ such that $E[g] > E[h]$.

(a) Suppose Elmo is risk-averse. Will Elmo necessarily prefer $g$ to $h$? Explain.

**Solution:** No. For example, let $u(w) = w^{\frac{1}{2}}$. Let $g = (0.25 \circ 32,\ 0.75 \circ 0)$ and $h = (0.5 \circ 15,\ 0.5 \circ 0)$. Then, $E[g] = 8 > \frac{15}{2} = E[h]$. But,

$$U(g) = 0.25\sqrt{32} \approx 1.41 < 1.94 \approx 0.5\sqrt{15} = U(h).$$

(b) What if Elmo is risk-neutral? Explain.

**Solution:** Yes. Since $u' > 0$,

$$U(g) = u[E(g)] > u[E(h)] = U(h).$$

(c) What if Elmo is risk-loving? Explain.

**Solution:** No. For example, let $u(w) = w^2$. Let $g = (1 \circ 5)$ and $h = (0.5 \circ 8,\ 0.5 \circ 0)$. Then, $E[g] = 5 > 4 = E[h]$. But,

$$U(g) = 5^2 = 25 < 32 = \frac{64}{2} = U(h).$$

4. Consider Tom and Jerry, who have identical utility function over money:

$$u_T(w) = u_J(w) = w^{\frac{1}{2}}.$$

Suppose each starts with initial wealth ¥$W$ on Sunday. On Monday, Tom borrows $W$ from Jerry. It is now Tuesday, and Tom is about to return $W$ to Jerry. Suppose you interrupt them and make the following “double-or-nothing” proposal:

“Instead of Tom paying Jerry back $W$ for sure, let’s flip this biased coin, which has probability $p$ of coming up ‘head’ and probability $1-p$ of coming up ‘tail’. If ‘head’ comes up then Tom pays $2W$ to Jerry, but if ‘tail’ comes up then Tom pays nothing.”

(a) What is the maximum value of $p$ for which Tom will accept this proposal? You may find the following useful: $\sqrt{0.5} = .707107$.

**Solution:** To find the maximum $p$ Tom will accept:

$$\begin{aligned}
p\,u_T(2W-2W)+(1-p)u_T(2W-0) &\geq u_T(W)\\
p\,(0)^{\frac{1}{2}}+(1-p)(2W)^{\frac{1}{2}} &\geq W^{\frac{1}{2}}\\
1-\frac{1}{\sqrt{2}} &\geq p\\
p_T &\leq 1-.707107 \approx 0.3.
\end{aligned}$$

(b) What is the minimum value of $p$ for which Jerry will accept this proposal?

**Solution:** To find the minimum $p$ Jerry will accept:

$$\begin{aligned} p\,u_J(2W)+(1-p)u_T(0) &\geq u_J(W) \\ p\,(2W)^{\frac{1}{2}}+(1-p)(0)^{\frac{1}{2}} &\geq W^{\frac{1}{2}} \\ p_J &\geq \frac{1}{\sqrt{2}} \approx .707107. \end{aligned}$$

(c) Compare the two values in parts (a) and (b). What does this mean?

**Solution:** $p_T < p_J$ means that the maximum value that Tom will accept is lower than the minimum that Jerry needs. Therefore, the proposal will not be accepted.

5. Consider a gamble based on a toss of a fair coin: you pay ¥$x$ and then toss the coin. If the toss results in a "head", you get $2x$, but if it results in a "tail", you get nothing. Suppose you have ¥200 in total and can make the bets in one of two ways. You can either bet the entire ¥200 on a single toss, or you can bet half of your money on the first toss and the remaining half on the second toss.

(a) If you are strictly risk-averse, which way would you prefer? You may assume $u(0)=0$.

**Solution:** Letting $g_1$ be the single-toss gamble and $g_2$ be the double-toss one, we have

$$\begin{aligned} EU(g_1) &= \frac{1}{2}u(400)+\frac{1}{2}u(0) = \frac{1}{4}u(200)+\frac{1}{4}u(200) \\ EU(g_2) &= \frac{1}{4}u(400)+\frac{1}{4}u(200)+\frac{1}{4}u(200)+\frac{1}{4}u(0) = \frac{1}{4}u(400)+\frac{1}{2}u(200). \end{aligned}$$

Since $u(\cdot)$ is strictly concave, we have

$$u(200) = u\left(\frac{1}{2}(400)+\frac{1}{2}(0)\right) > \frac{1}{2}u(400)+\frac{1}{2}u(0) = \frac{1}{2}u(400).$$

Therefore $EU(g_2) > EU(g_1)$.

(b) If you are risk-neutral, which way would you prefer?

**Solution:** Since risk neutrality means $EU(g) = U(E[g])$, and $E[g_1] = 200 = E[g_2]$, you would be indifferent between the two gambles.

6. Consider an individual with initial wealth ¥$W$ and utility function over money given by $u(w) = w^{\frac{1}{2}}$. The individual faces possible loss of ¥$L$, where $L$ is either $W$, $\frac{W}{2}$, or 0, each with probability $\frac{1}{3}$. Suppose an insurance that will cover her entire loss is available at price ¥$p$. That is, for the total price ¥$p$, the insurance company will make a payment equal to the amount of the realized loss.

   (a) Find the individual's expected utility when she does not buy the insurance.
   **Solution:**

$$\begin{aligned} U_{no} &= \frac{1}{3}W^{\frac{1}{2}} + \frac{1}{3}\left(\frac{W}{2}\right)^{\frac{1}{2}} + \frac{1}{3}0^{\frac{1}{2}} \\ &= \frac{1+\sqrt{2}}{3\sqrt{2}}W^{\frac{1}{2}} \approx (0.569)W^{\frac{1}{2}}. \end{aligned}$$

   (b) Find the individual's expected utility if she buys the insurance.
   **Solution:**

$$U_{yes} = (W-p)^{\frac{1}{2}}.$$

   (c) Let $W = 100$. What is the maximum price at which she will buy the insurance?

**Solution:** We need

$$\begin{aligned} (W-p)^{\frac{1}{2}} &\geq \frac{1+\sqrt{2}}{3\sqrt{2}}W^{\frac{1}{2}} \\ 100 - \left(\frac{1+\sqrt{2}}{3\sqrt{2}}\right)^2(100) &\geq p \\ p &\leq 100-(0.569)^2(100) = 67.62. \end{aligned}$$

7. Consider the insurance example given in the class. Let $\alpha$ be the probability that the earthquake will occur. Suppose now that the price of insurance is given by $p$ and that $p > \alpha$.

   (a) Set up the consumer's expected utility maximization problem. Assume interior solution and derive the first order condition.
   **Solution:** The consumer solves

$$\max_x \ \alpha u(w-px-L+x) + (1-\alpha)u(w-px).$$

   The first order condition is

$$\alpha u'(w-px-L+x)(1-p) + (1-\alpha)u'(w-px)(-p) = 0$$

(b) Show whether a strictly risk averse consumer still fully insures.

**Solution:** Rearranging the first order condition yields

$$\begin{aligned} u'(w-px-L+x) &= \frac{p(1-\alpha)}{\alpha(1-p)}u'(w-px) \\ \Rightarrow \quad u'(w-px-L+x) &> u'(w-px) \\ \Rightarrow \quad w-px-L+x &< w-px \\ \Rightarrow \quad x &< L. \end{aligned}$$

8. Consider again the insurance model discussed in the lecture. Assuming that the price of insurance is fair, how much insurance will a *strictly risk-loving* individual buy? (Please remember to pay attention to the second order condition. You may assume that $u'' > 0$.)

**Solution:** DM's expected utility maximization problem is

$$\max_{0\le x\le L} U(g(x))$$

$$\iff \quad \max_{0\le x\le L} au(W-ax-L+x)+(1-a)u(W-ax).$$

We first find the critical values by solving the first order condition with equality:

$$\begin{aligned} \frac{dU(g(x))}{dx} &= a(1-a)u'\left(W-ax^*-L+x^*\right)-a(1-a)u'\left(W-ax^*\right) \\ &= 0. \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad u'\left(W-ax^*-L+x^*\right) &= u'\left(W-ax^*\right) \\ W-ax^*-L+x^* &= W-ax^* \quad \text{since } u' \text{ is increasing} \\ x^* &= L. \end{aligned}$$

However, as we show below, $x^* = L$ does not satisfy the second order condition:

$$\begin{aligned} \frac{d^2U(g(x))}{dx^2} &= a(1-a)^2u''\left(W-ax^*-L+x^*\right)+a^2(1-a)u''\left(W-ax^*\right) \\ &> 0 \quad \text{since } u'' > 0. \end{aligned}$$

So, $x^* = L$ is a utility minimizer not a maximizer. The only remaining possibility for a maxima is the left boundary $x^{**} = 0$. We check to see if it satisfies the first order condition for the left boundary:

$$\begin{aligned} \left.\frac{dU(g(x))}{dx}\right|_{x^{**}=0} &= a(1-a)u'(W-L)-a(1-a)u'(W) \\ &< 0 \quad \text{since } W-L<W \text{ and } u''>0. \end{aligned}$$

So, the first order condition is satisfied and the solution is $x^{**} = 0$.

9. Consider an individual with initial wealth $W = \$1,000$ and utility function over money given by $u(w) = w^{\frac{1}{2}}$. The individual faces loss of $L = \$800$ with probability $\frac{1}{4}$. Suppose an insurance is available at price $\$p$ per unit, where one unit of insurance pays the insured \$1 if loss occurs and \$0 otherwise.

   (a) What is the fair price for this insurance, and how much insurance will the individual buy at this price?

   **Solution:** Fair price is the probability of loss $p = 0.25$. Since the individual is strictly risk averse, she will fully insure. That is optimal purchase of insurance is $x^* = 800$.

   (b) Suppose $p = \$0.3$. How much insurance will the individual buy?

   **Solution:** Letting $g(x)$ be the gamble that results from buying $x$ units of insurance, the individual solves

$$\begin{aligned}\max_x U(g(x)) \iff & \max_x \quad \frac{1}{4}u(W - px - L + x) + \frac{3}{4}u(W - px) \\ \iff & \max_x \quad \frac{1}{4}\big(W - L + (1-p)x\big)^{\frac{1}{2}} + \frac{3}{4}\big(W - px\big)^{\frac{1}{2}}\end{aligned}$$

   Solving the first order condition yields

$$\begin{aligned}U'(g(x)) = 0 \iff & \frac{1}{4}\left(\frac{1}{2}\right)\big(W - L + (1-p)x\big)^{-\frac{1}{2}}(1-p) + \frac{3}{4}\left(\frac{1}{2}\right)\big(W - px\big)^{-\frac{1}{2}}(-p) = 0 \\ \iff & \big(W - L + (1-p)x\big)^{-\frac{1}{2}}(1-p) = 3p\big(W - px\big)^{-\frac{1}{2}} \\ \iff & \big(W - L + (1-p)x\big)(1-p)^{-2} = (3p)^{-2}\big(W - px\big) \\ \iff & \big(1000 - 800 + (1-0.3)x\big)(1-0.3)^{-2} = (3 \times 0.3)^{-2}\big(1000 - 0.3x\big) \\ \iff & \big(200 + 0.7x\big)\left(\frac{0.9}{0.7}\right)^2 = \big(1000 - 0.3x\big) \\ \iff & \big(200 + 0.7x\big)\left(\frac{0.9}{0.7}\right)^2 = \big(1000 - 0.3x\big) \\ \iff & \big(200 + 0.7x\big)1.65 = \big(1000 - 0.3x\big) \\ \iff & 330 + 1.16x = 1000 - 0.3x \\ \iff & x^* = \frac{1000 - 330}{1.46} = \frac{670}{1.46} = 458.90.\end{aligned}$$

   Note that since the price is higher than the fair price, the individual underinsures. Finally, to insure that the critical value $x^*$ is indeed the (expected) utility maximizer, we should verify that the second order condition is satisfied at the critical value $x^*$ (that is, $U'' < 0$ at $x^*$). This holds because the individual is strictly risk averse.

   (c) Suppose $p = \$0.3$, but now the individual's utility function over money is $u(w) = w^2$. How much insurance will the individual buy? As clarified in the class, assume that the individual cannot insure more than the amount of loss.

   **Solution:** Since the individual is risk-loving, the second order condition for maximization ($U'' < 0$) does not hold at the critical value. In fact, we have

$$
\begin{aligned}
U(g(x)) &= \frac{1}{4}\big(W-L+(1-p)x\big)^2+\frac{3}{4}\big(W-px\big)^2\\
U'(g(x)) &= \frac{2}{4}\big(W-L+(1-p)x\big)(1-p)+\frac{6}{4}\big(W-px\big)(-p)\\
U''(g(x)) &= \frac{2}{4}(1-p)^2+\frac{6}{4}(-p)^2>0.
\end{aligned}
$$

So $U$ is convex. This means the (expected) utility maximizing level of insurance occurs at the boundary, $x^*=0$. Note that if the individual buys zero insurance, her expected utility is

$$
\begin{aligned}
U(g(0)) &= \frac{1}{4}\big(W-L\big)^2+\frac{3}{4}\big(W\big)^2=\frac{1}{4}(200)^2+\frac{3}{4}(1000)^2\\
&= \frac{1}{4}(40,000)+\frac{3}{4}1000,000=10,000+750,000=760,000,
\end{aligned}
$$

which is higher than if the individual buys $L$ amount of insurance:

$$
U(g(L))=\frac{1}{4}\big(W-pL\big)^2+\frac{3}{4}\big(W-pL\big)^2=(1000-0.3(800))^2=760^2=577,600.
$$

10. Consider again the insurance model discussed in class. Let $\alpha$ be the probability that an earthquake will occur and $1-\alpha$ the probability that it will not. Let $u(\cdot)$ be the consumer's utility function over money.

    Letting $w_1$ be the final amount of wealth the consumer ends ups with if an earthquake occurs and $w_2$ be the final wealth if it does not, we can denote the consumer's expected utility as $U(w_1,w_2)$.

    (a) Express $U(w_1,w_2)$ in terms of $\alpha$, $w_1$, $w_2$, and $u$.

    **Solution:**

    $$U(w_1,w_2) \;=\; \alpha u(w_1)+(1-\alpha)u(w_2)$$

    (b) Since $\alpha$ is fixed, we can think of $U(w_1,w_2)$ as a utility function over wealth bundle $(w_1,w_2)$. Find the marginal rate of substitution between $w_1$ and $w_2$.

    **Solution:** We have

    $$MRS \;=\; \frac{\frac{\partial U}{\partial w_1}}{\frac{\partial U}{\partial w_2}} \;=\; \frac{\alpha u'(w_1)}{(1-\alpha)u'(w_2)}.$$

    (c) Argue that the indifference curves are convex if the individual is risk averse.

    **Solution:** We have

    $$\frac{\partial\, MRS}{\partial w_1} \;=\; \frac{\alpha u''(w_1)\left[(1-\alpha)u'(w_2)\right]-\alpha u'(w_1)\left[(1-\alpha)u''(w_2)\frac{dw_2}{dw_1}\right]}{[(1-\alpha)u'(w_2)]^2} \;<\; 0.$$

This implies that the indifference curve is getting flatter as $w_1$ increases, which means it is convex.

11. Consider an investor whose utility function over money is

$$u(w) = 2w^{\frac{1}{2}}.$$

The investor can invest in a riskless asset that returns 1 (gross return per ¥1 invested) for sure, or a risky asset that returns 1.4 with probability $\frac{3}{4}$ and 0.8 with probability $\frac{1}{4}$.

(a) Suppose the investor's initial wealth is ¥1000. Letting $x$ denote the amount invested in the risky asset, write the investor's expected utility as a function of $x$.

**Solution:** We have

$$g(x) = \begin{cases} 1000 - x + 1.4x = 1000 + 0.4x & \text{with probability } \frac{3}{4} \\ 1000 - x + 0.8x = 1000 - 0.2x & \text{with probability } \frac{1}{4} \end{cases}$$

So,

$$\begin{aligned} U(g(x)) &= \frac{3}{4}u(1000 + 0.4x) + \frac{1}{4}u(1000 - 0.2x) \\ &= \frac{3}{4}\left(2(1000 + 0.4x)^{\frac{1}{2}}\right) + \frac{1}{4}\left(2(1000 - 0.2x)^{\frac{1}{2}}\right) \end{aligned}$$

(b) Find the optimal amount to invest in the risky asset.

**Solution:** Investor solves:

$$\max_x \frac{3}{4}\left(2(1000 + 0.4x)^{\frac{1}{2}}\right) + \frac{1}{4}\left(2(1000 - 0.2x)^{\frac{1}{2}}\right)$$

FOC is given by

$$\left(\frac{3}{4}\right)\left(\frac{4}{10}\right)(1000 + 0.4x)^{-\frac{1}{2}} - \left(\frac{1}{4}\right)\left(\frac{2}{10}\right)(1000 - 0.2x)^{-\frac{1}{2}} = 0$$

$$\begin{aligned} \Rightarrow \left(\frac{3}{10}\right)(1000 + 0.4x)^{-\frac{1}{2}} &= \left(\frac{1}{20}\right)(1000 - 0.2x)^{-\frac{1}{2}} \\ \Rightarrow \left(\frac{10}{3}\right)^2 (1000 + 0.4x) &= (20)^2(1000 - 0.2x) \\ \Rightarrow x &= \frac{(400)(1000) - \left(\frac{100}{9}\right)(1000)}{\left(\frac{100}{9}\right)(0.4) + (400)(0.2)} \approx 4605.263. \end{aligned}$$

Note that $x^* = 4605.263$ is greater than the initial wealth. So, if so-called "short sale" is possible, the investor will borrow additional $\$3,605.263$ to invest in the risky asset. If borrowing is not possible, then the investor will put the maximum amount in the risky asset. That is, $x^* = 1000$.