Intermediate Microeconomics

Exercises: Consumer Theory

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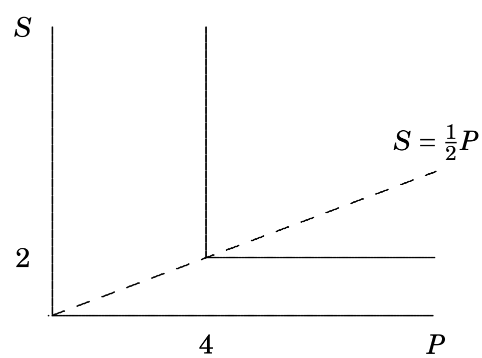
1 Consumer Theory

1. Suppose Pizza (P) and Soda (S) are perfect complements to Ling Ling. She always consumes them at the ratio of 4 slices of Pizza to 2 cups of Soda.

   (a) Placing Pizza on the horizontal axis and Soda on the vertical axis, sketch the indifference curve corresponding to utility level 4.

   Solution: Using the utility function in Part(b), the sketch of the indifference curve corresponding to utility level 4 is:



   (b) Write Ling Ling's utility function.

   Solution: For every slice of Pizza, Ling Ling needs to drink 1/2 cups of soda. So, the "kink" point of her indifference curve must satisfy $S = \frac{1}{2}P$. Therefore, we can use the utility function

$$u(P,S) = \min\{P, 2S\}.$$

   Note that any strictly increasing transformation of this utility function, $u(P,S) = \min\{2P, 4S\}$ for example, will also work.

   (c) Find the marginal rate of substitution (MRS) of Soda for Pizza for this utility function. Given an interpretation of your result.

   Solution: For any point (P,S) on the "kink" ($2S = P$), MRS is undefined since the indifference curve is not differentiable at these points. For any point on the"horizontal portion" ($P > 2S$) of the indifference curve, MRS = 0. Since additional consumption of Pizza without corresponding increase in Soda does not increase Ling Ling's utility, she is not willing to trade any Soda for Pizza. For any point on the "vertical portion" ($2S > P$), MRS = ∞. Since marginal reduction in the consumption of Pizza does not decrease Ling Ling's utility, she is willing to trade soda for any amount of Pizza.

2. Consider the following two utility functions:

$$\begin{aligned} (1) \quad u(x_1,x_2) &= x_1^{\frac{1}{4}} x_2^{\frac{3}{4}} \\ (2) \quad u(x_1,x_2) &= \ln x_1 + 3\ln x_2 \end{aligned}$$

   (a) For each of the above two utility functions, graph the indifference curve going through bundle $(2,2)$ and also one through bundle $(4,4)$. Indicate the utility value associated with each indifference curve.

   Solution: The two graphs has exactly the same indifference curves.

   (b) Find the Marshallian demand for each utility function.

   Solution: For $u(x_1,x_2) = x_1^{\frac{1}{4}} x_2^{\frac{3}{4}}$, we have the following.

$$MRS = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\frac{1}{4}x_1^{-\frac{3}{4}} x_2^{\frac{3}{4}}}{\frac{3}{4}x_1^{\frac{1}{4}} x_2^{-\frac{1}{4}}} = \frac{x_2}{3x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = \frac{3p_1}{p_2}x_1$$

Substituting this into the budget equation yields:

$$p_1x_1 + p_2\left(\frac{3p_1}{p_2}x_1\right) = I \quad \Rightarrow \quad 4p_1x_1 = I.$$

So,

$$x_1(p_1,p_2,I) = \frac{I}{4p_1} \quad \text{and} \quad x_2(p_1,p_2,I) = \frac{3p_1}{p_2}x_1(p_1,p_2,I) = \frac{3I}{4p_2}.$$

For $u(x_1,x_2) = \ln x_1 + 3\ln x_2$, we have

$$MRS = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\frac{1}{x_1}}{\frac{3}{x_2}} = \frac{x_2}{3x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = \frac{3p_1}{p_2}x_1$$

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So,

$$x_1(p_1,p_2,I) = \frac{I}{4p_1} \quad \text{and} \quad x_2(p_1,p_2,I) = \frac{3p_1}{p_2}x_1(p_1,p_2,I) = \frac{3I}{4p_2}.$$

3. Suppose pizza (P) and hamburgers (H) are perfect substitutes for Ling Ling. She is always willing to substitute two pizzas for one hamburger.

   (a) Letting good 1 (horizontal good) be Pizza and good 2 (vertical good) be hamburger, derive Ling Ling's utility function.

   Solution: Ling Ling being willing to substitute one two pizzas for one hamburger means she is willing to substitute $\frac{1}{2}$ hamburger (good 2) for 1 pizza (good 1). We have

$$MRS = -\frac{dx_H}{dx_P} = \frac{1}{2} \quad \text{and} \quad MRS = \frac{MU_P}{MU_H}.$$

   So, $\frac{MU_P}{MU_H} = \frac{1}{2}$. So, $u(x_P, x_H) = x_P + 2x_H$ represents Ling Ling. (Actually, any $u(x_P, x_H) = ax_P + 2ax_H$, where $a > 0$ also works).

   (b) Let p_P be the price of pizza and p_H be the price of hamburger. Find the demand for pizza and hamburger when $p_P = 1$ and $p_H = 2$.

   Solution: For x^* to be an interior solution, x^* must satisfy (1) MRS = price ratio and (2) the budget equation.

$$MRS = \frac{\frac{\partial u}{\partial x_P}}{\frac{\partial u}{\partial x_H}} = \frac{p_P}{p_H} \quad \Rightarrow \quad \frac{1}{2} = \frac{1}{2}.$$

   Since this is always true, MRS = price ratio condition places no restriction on the solution. So any bundle that satisfies the budget equality is a solution. I.e., any $x^* = (x_P^*, x_H^*)$ such that $x_P^* + 2x_H^* = I$ is a solution.

   (c) Let p_P be the price of pizza and p_H be the price of hamburger. Find the demand for pizza and hamburger when $p_P = 1$ and $p_H > 2$.

   Solution: Now,

$$MRS = \frac{\frac{\partial u}{\partial x_P}}{\frac{\partial u}{\partial x_H}} = \frac{1}{2} > \frac{p_P}{p_H}.$$

   Since this is never true, there is no interior solution. The above inequality implies that

$$\frac{\frac{\partial u}{\partial x_P}}{p_P} > \frac{\frac{\partial u}{\partial x_H}}{p_H},$$

   so the consumer always wants to shift purchase away from hamburger towards pizza. So the solution occurs where this cannot happen any more. That is, the solution is $x_P^* = \frac{I}{p_P} = I$ and $x_H^* = 0$. You can also show this graphically.

   (d) Let p_P be the price of pizza and p_H be the price of hamburger. Find the demand for pizza and hamburger when $p_P = 1$ and $p_H < 2$.

   Solution: Now,

$$MRS = \frac{\frac{\partial u}{\partial x_P}}{\frac{\partial u}{\partial x_H}} = \frac{1}{2} < \frac{p_P}{p_H}.$$

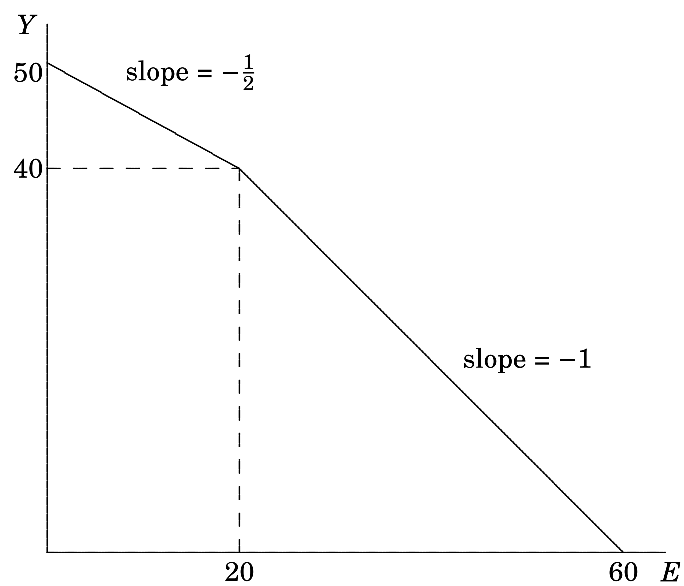
Since this is also never true, there is no interior solution. In fact, since

$$\frac{\frac{\partial u}{\partial x_P}}{p_P} < \frac{\frac{\partial u}{\partial x_H}}{p_H},$$

the consumer always wants to shift purchase away from pizza towards hamburger, so the solution occurs where this cannot happen any more. That is, the solution is $x_P^* = 0$ and $x_H^* = \frac{I}{p_H}$. Again, you may show this graphically.

4. There are two goods in the economy: electricity (E) and the composite good (Y). Suppose the government subsidizes the first 20kWh of electricity consumption so that the price of electricity is ¥5 for first 20kWh and ¥10 thereafter, while the price of the composite good remains constant at p_Y = ¥10.

   (a) Placing the composite good on the vertical axis and electricity on the horizontal axis, carefully sketch the budget set for the consumer who has income ¥500.



   (b) Suppose a consumer's utility function is given by $u(E,Y) = EY$. Find the consumer's utility maximizing bundle.
   Solution: We have

$$MRS = \frac{\frac{\partial u}{\partial E}}{\frac{\partial u}{\partial Y}} = \frac{Y}{E}$$

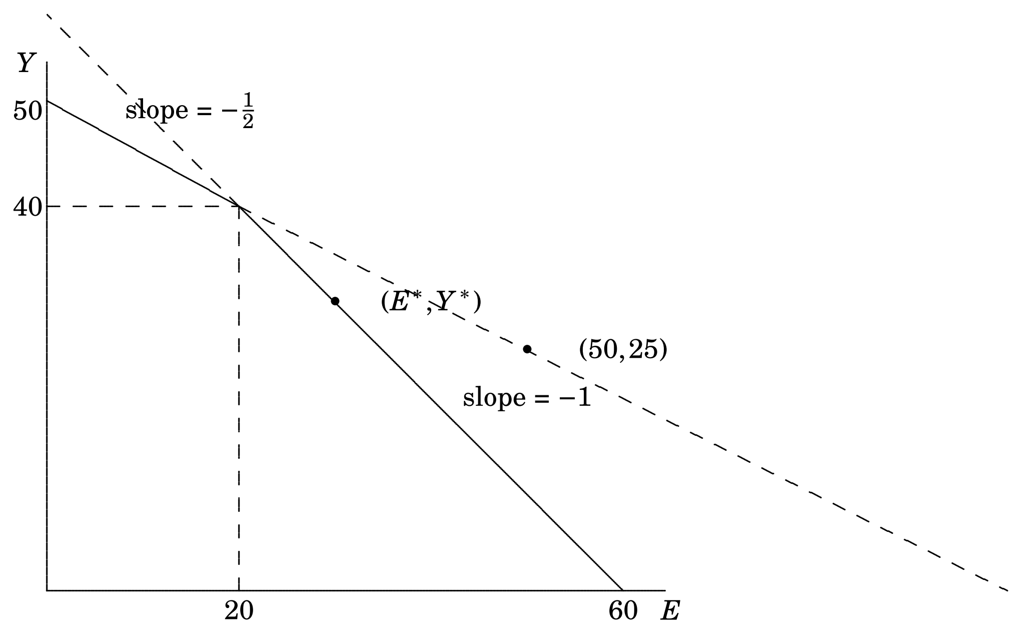
Solution occurs where $MRS = 1$ or $MRS = \frac{1}{2}$. Letting $MRS = 1$ yields

$$Y = E \quad \Rightarrow \quad 10E + 10E = 600 \Rightarrow E = Y = 30.$$

But, letting $MRS = \frac{1}{2}$ yields

$$2Y = E \quad \Rightarrow \quad 5(2Y) + 10Y = 500 \Rightarrow Y = 25,\ E = 50,$$

which does not satisfy the actual budget equality. So, the solution must be $E^* = 30$, $Y^* = 30$. See the picture below.



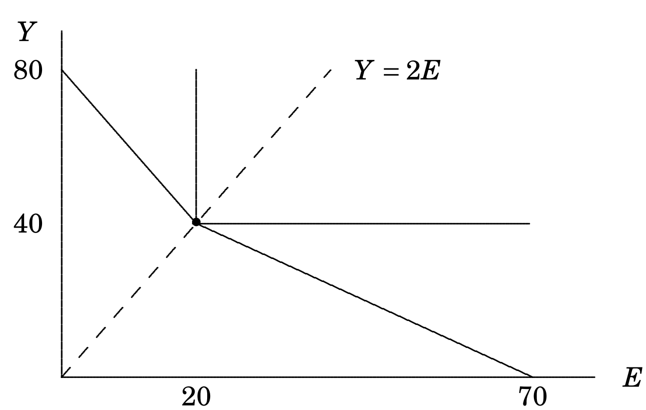
5. Consider again the budget set example involving electricity and the composite good that was discussed in class.

   (a) Suppose, instead of subsidizing the first 20kWh of electricity consumption, the government wants to encourage a greater use of electricity by subsidizing the consumption beyond the first 20kWh. In particular, suppose the price of electricity is ¥10 for first 20kWh and ¥4 thereafter, while the price of the composite good remains the same at $p_y = ¥5$. Carefully sketch the budget set for the consumer who has income ¥400.

   Solution: By solving

$$10(20) + 5y = 400 \quad \Rightarrow \quad y = 40,$$

   we see that the “kink” on the budget line occurs at $x = 20$ and $y = 40$.



(b) Suppose a consumer's utility function over electricity (E) and the composite good (Y) is given by:

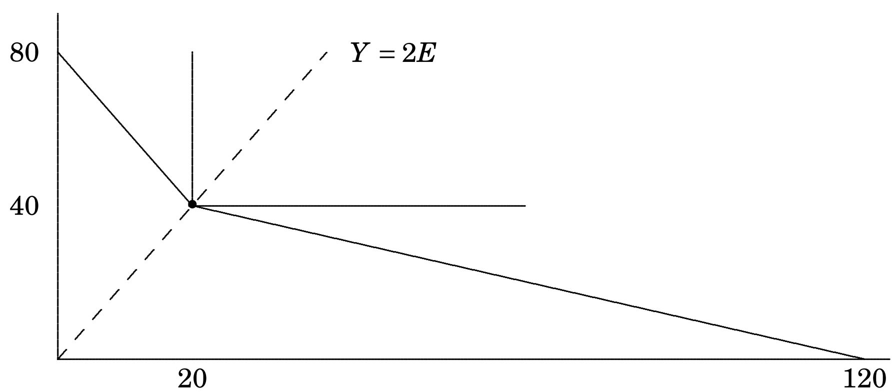
$$u(E,Y) = \min\{2E, Y\}.$$

Find the consumer's utility maximizing bundle.

Solution: Utility maximizing bundle occurs where the "kink" equation, $Y = 2E$, intersects with the budget line. We see from the graph in Part (a) that the solution is $E^* = 20$ and $Y^* = 40$.

(c) Will the consumer consume more electricity if the price of electricity beyond the first 20kWh is lowered to ¥2? Explain.

Solution: As seen below, lower price of electricity will have no effect since the budget line pivots around the kink. This is a consequence of the fact that the consumer has a Leontief preference.



6. Suppose there are two goods in the economy: Food (F) and Composite good (Y), with $P_F = 1$ and $P_Y = 2$. Consider a consumer who has utility function

$$u(F,Y) = \ln F + \ln Y$$

and income $I = \$100$.

(a) Find the consumer's consumption choice.

Solution: This utility function is a transformation of standard Cobb-Douglas utility function $F^{\frac{1}{2}}Y^{\frac{1}{2}}$. Thus, the optimal choice is given by

$$F^* = \frac{I}{2P_F} = \frac{100}{2(1)} = 50 \quad \text{and} \quad Y^* = \frac{I}{2P_Y} = \frac{100}{2(2)} = 25.$$

(b) Suppose, to promote better nutrition, the government institutes a welfare program that gives every consumer with income less than \$500 additional \$100. Find the consumer's new consumption choice. How much of the government's payment went into purchasing Food?

Solution: Now, the optimal choice is given by

$$F^* = \frac{I'}{2P_F} = \frac{100+100}{1(2)} = 100 \quad \text{and} \quad Y^* = \frac{I'}{2P_Y} = \frac{100+100}{2(2)} = 50.$$

The consumer bought 50 additional units of Food. Thus, \$50 of the government support went into purchasing food.

(c) Suppose rather than giving cash, the government gives a voucher that can only be used to purchase (up to) \$100 worth of Food. How does this affect the consumer's consumption choice?

Solution: By using \$100 voucher to pay for 100 units of food and using \$100 cash income to buy 50 units of the consumption good, the consumer can achieve exactly the same bundle as in Part(b). Since this is optimal in the absence of restriction on the goods that can be purchased, it must be optimal with the restriction as well. Therefore, $F^* = 100$ and $Y^* = 50$ is still the solution.

(d) Which of the two programs is more effective in achieving the government's goal? Explain.

Solution: The two programs have exactly the same effect. Giving food voucher will induce the consumer to buy more food than giving cash only if the consumer was buying less food than the amount of the government's cash support. Otherwise, the voucher will only shift the method by which the consumer pays for food.

7. Consider a consumer whose utility function over good 1 and good 2 is given by

$$u(x_1, x_2) = x_2 - x_1^2.$$

(a) Sketch the indifference curve corresponding to utility level 1.

Solution:

(b) Is the indifference curve convex? Is it monotone?

Solution: The indifference is convex but not monotone. For example, $u(1,1) = 0 > -2 = u(2,2)$.

(c) Suppose the consumer has income $I = 100$ and faces prices $p_1 = 1$ and $p_2 = 2$. Find the consumer's optimal consumption bundle.

Solution: Note that the consumer's utility is decreasing in the consumption of good 1. That is, consumption of good 1 actually makes the consumer worse off. Thus, the consumer's utility maximizing choice would be to buy zero amount of good 1:

$$x_1^* = 0 \qquad \text{and} \qquad x_2^* = \frac{I}{p_2} = \frac{100}{2} = 50.$$

(d) For what prices will the consumer consume strictly positive amount of both goods? Interpret this result.

Solution: An interior solution must satisfy

$$MRS = \frac{-2x_1}{1} = \frac{p_1}{p_2}.$$

So, one of the prices must be negative. If p_1 is positive and p_2 is negative, then the consumer will want to buy zero amount of good 1 and infinite amount of good 2. Therefore, the only possibility for an interior solution is to have p_1 negative and p_2 positive. That is, since good 1 is actually a "bad" the consumer needs to be paid (compensated) to consume it.

8. Suppose Pizza (P) and Hamburgers (H) are perfect substitutes to Ling Ling. Two slices of Pizza always give her the same utility level as one Hamburger.

   (a) Express Ling Ling's preference with a utility function.

   Solution: Ling Ling's utility function is given by

$$U(P,H) = aP + bH, \quad \text{where } \frac{a}{b} = \frac{1}{2}.$$

   For example, $U(P,H) = P + 2H$.

   (b) Find Ling Ling's optimal consumption bundle.

   Solution: Let p_p and p_h be the price of Pizza and Hamburger, respectively. Since

$$MRS = \frac{\frac{\partial U}{\partial P}}{\frac{\partial U}{\partial H}} = \frac{1}{2},$$

   the demand function is given by

$$(x_p^*, x_h^*) = \begin{cases} (\frac{I}{p_p}, 0) & \text{if } \frac{p_p}{p_h} < \frac{1}{2} \\ \text{any } (P,H) \text{ s.t. } p_p P + p_h H = I & \text{if } \frac{p_p}{p_h} = \frac{1}{2} \\ (0, \frac{I}{p_h}) & \text{if } \frac{p_p}{p_h} > \frac{1}{2} \end{cases}.$$

9. Pat's preference is given by

$$u(x_1, x_2) = \min\{x_1, x_2\}.$$

   Currently, prices are $p = (p_1, p_2)$ and Pat's income is I. Is he better off if the price of good one is halved so that $p = (\frac{p_1}{2}, p_2)$, or if his income is doubled to $2I$?

   Solution 1: Recall that $u(x_1, x_2)$ is a Leontieff utility function which has L-shaped indifference curves. The corner points of the indifference curves satisfy $x_1 = x_2$. So, the optimal consumption bundle satisfies

$$\begin{array}{ll} (1) & x_1 = x_2, \quad \text{and} \\ (2) & p_1 x_1 + p_2 x_2 = I. \end{array}$$

   To solve, substitute (1) into (2) to obtain:

$$p_1 x_1 + p_2 x_1 = I,$$

which gives

$$x_1 = \frac{I}{p_1 + p_2}.$$

So, the Marshallian demand function is

$$x_1(p, I) = x_2(p, I) = \frac{I}{p_1 + p_2}.$$

Letting $v(p_1, p_2, I)$ denote the maximum utility that is achievable at (p_1, p_2, I), we have

$$\begin{aligned} \min\{x_1(p,I), x_2(p,I)\} &= \min\left\{\frac{I}{p_1+p_2}, \frac{I}{p_1+p_2}\right\} \\ &= \frac{I}{p_1+p_2}. \end{aligned}$$

Since

$$\frac{I}{\frac{1}{2}p_1 + p_2} < \frac{I}{\frac{1}{2}p_1 + \frac{1}{2}p_2} = \frac{2I}{p_1 + p_2},$$

Pat is better off if the prices remain the same but her income is doubled. Note that this can also be solved quite easily graphically.

10. A firm that is located in country H, where price levels are $p = (1, 2)$, needs to send one of its two employees to its branch in country F. However, in country F price levels are $p' = (3, 1)$, so the firm will have to pay additional salary to ensure that its employee is equally well-off in country F as she was in country H. Suppose the utility functions of the two employees are

$$u_1(x_1, x_2) = \ln x_1 + \ln x_2 \quad \text{and} \quad u_2(x_1, x_2) = x_1 + x_2.$$

The two employees are otherwise identical, including current salary. If the firm wants to minimize the additional salary it needs to pay, which employee should it send? Explain.

Solution: Letting I be the employees' current salary, the demand function for the first employee is given by

$$(x_1(p, I)\ ,\ x_2(p, I)) = \left(\frac{I}{2(p_1)}\ ,\ \frac{I}{2(p_2)}\right),$$

and her utility at this consumption bundle is

$$u_1(x_1(p,I), x_2(p,I)) = \ln\left(\frac{I}{2(p_1)}\right) + \ln\left(\frac{I}{2(p_2)}\right) = \ln\left(\frac{I^2}{4p_1p_2}\right).$$

The new salary $\hat{I}_1$ firm would have to pay her must make her indifferent between the two countries. That is

$$\ln\left(\frac{\hat{I}_1^2}{12}\right) = \ln\left(\frac{I^2}{8}\right) \quad \Rightarrow \quad \hat{I}_1 = \sqrt{\frac{12}{8}I^2} = \left(\frac{3}{2}\right)^{\frac{1}{2}} I.$$

Since $p_1 < p_2$, the second employee's demand function in country H is given by

$$(x_1(p,I)\ ,\ x_2(p,I)) = \left(\frac{I}{p_1}\ ,\ 0\right),$$

and her utility at this consumption bundle is

$$u_2(x_1(p,I), x_2(p,I)) = \frac{I}{p_1} = I.$$

In contrast, $p'_1 > p'_2$ in country F. Thus, her demand function in country is given by by

$$\left(x_1(p', \hat{I}_2)\ ,\ x_2(p', \hat{I}_2)\right) = \left(0\ ,\ \frac{\hat{I}_2}{p'_2}\right),$$

and her utility at this consumption bundle is

$$u_2\left(x_1(p', \hat{I}_2), x_2(p', \hat{I}_2)\right) = \frac{\hat{I}_2}{p'_2} = \hat{I}_2.$$

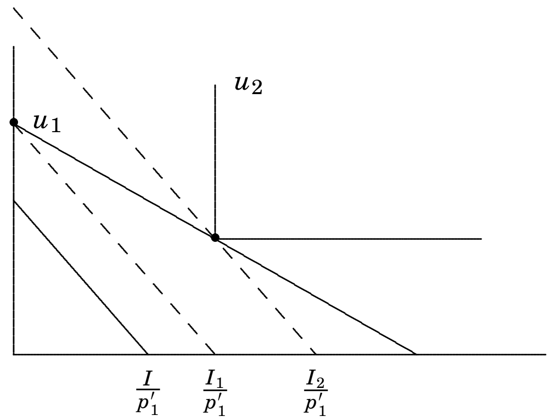
Thus, the firm will have to pay her $\hat{I}_2 = I$. Since $\hat{I}_2 = I < I\sqrt{\frac{3}{2}} = \hat{I}_1$, the firm should send employee 2.

11. A firm that is located in country H, where price levels are $p = (1,1)$, needs to send one of its two employees to its branch in country F. However, in country F price levels are $p' \gg p$, so the firm will have to pay additional salary to ensure that its employee is equally well-off in country F as she was in country H. Suppose the utility functions of the two employees are

$$u_1(x_1, x_2) = x_1 + x_2 \quad \text{and} \quad u_2(x_1, x_2) = \min\{x_1, x_2\}.$$

The two employees are otherwise identical, including current salary. If the firm wants to minimize the additional salary it needs to pay, which employee should it send? Explain.

Solution: As seen in the graph below, when $p'_1 > p'_2$, the amount of money required to compensate employee 1 is less than that of employee 2. This is because employee 1 can substitute away from more expensive good towards the cheaper good while employee 2's utility function does not allow any substitution. It's easy to see that same holds when $p'_1 < p'_2$. When $p'_1 = p'_2$ then both employees require same amount of compensation.



12. Steve's utility function over leisure and consumption is given by

$$u(L,Y) = \min\{3L, Y\}.$$

Wage rate is w and the price of the composite consumption good is $p = 1$.

(a) Suppose $w = 5$. Find the optimal leisure - consumption combination. What is the amount of hours worked?

Solution: The solution occurs where both the "kink" equation and the budget constraint equation, $Y = w(24-L)$, is satisfied.

$$\begin{aligned} 3L &= Y \\ wL + Y &= 24w \\ \Rightarrow \quad 5L + (3L) &= 24(5) \\ L^* &= \frac{120}{8} = 15 \\ Y^* &= 3L^* = 45 \\ \text{hours worked} &= 9. \end{aligned}$$

(b) Suppose the overtime law is passed so that every worker needs to be paid 1.5 times their current wage for hours worked beyond the first 8 hours. Will this law induce Steve to work more hours? If so, how many? If not, explain.

Solution: Since Steve is already working more than 8 hours, the overtime law will have an effect:

Steve's budget equation is now $Y = 8w + 1.5w(24-8-L)$. So,

$$\begin{aligned} 3L &= Y \\ 1.5wL + Y &= 8w + 1.5 * 16w \\ \Rightarrow \quad 7.5L + (3L) &= 8(5) + 16(7.5) \\ L^{**} &= \frac{160}{10.5} = 15.24 \\ Y^{**} &= 3L^{**} = 45.72 \\ \text{hours worked} &= 8.76. \end{aligned}$$

So, Steve will work 0.24 less hours. While the introduction of overtime wage increases the opportunity cost of leisure, it also raises Steve's real income. In this example, income effect dominates the substitution effect, and, as a result, he purchases both more leisure and consumption good than before.

13. Consider a consumer that lives for two periods. The consumer's utility function over the consumption in the two periods is given by

$$u(x_1, x_2) = x_1 x_2.$$

Suppose she receives income I_1 in period 1 but nothing in period 2. However, the consumer can borrow or lend freely at the gross interest rate $(1+r)$.

(a) Set up the consumer's utility maximization problem.

Solution: Consumer solves

$$\max_{x_1,x_2} u(x_1,x_2) \quad \text{s.t. } (1+r)x_1 + x_2 = (1+r)I_1.$$

(b) Letting $I_1 = 100$ and $r = 0.1$, find the optimal consumption choice and savings. What utility level does the consumer achieve?

Solution: Since $x_1 x_2$ and $x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ have the same demand function, we have

$$\begin{aligned} x_1(r, I_1) &= \frac{(1+r)I_1}{2(1+r)} = \frac{I_1}{2} = 50 \\ x_2(r, I_1) &= \frac{(1+r)I_1}{2} = \frac{(1.1)(100)}{2} = 55 \\ s(r, I_1) &= I_1 - x_1(r, I_1) = 100 - 50 = 50 \\ u(x_1(r, I_1), x_1(r, I_1)) &= 50(55) = 2750. \end{aligned}$$

(c) Suppose the interest rate goes up to $r' = 0.2$. How does the period 1 consumption change? Decompose the total effect into substitution effect and income effect.

Solution: Since $x_1(r, I_1)$ does not depend on r, period 1 consumption will not change. That is, the total effect on good 1 is zero. To determine the substitution and income effect, first find the Hicksian demand by solving the EMP:

$$\min_{x_1,x_2} (1+r')x_1 + x_2 \quad \text{s.t.} \quad x_1 x_2 = u.$$

$$
\begin{aligned}
\frac{x_2}{x_1} &= \frac{1+r'}{1} \\
\Rightarrow \quad x_2 &= (1+r')x_1
\end{aligned}
$$

substitute into the utility constraint:

$$
\begin{aligned}
\Rightarrow \quad x_1((1+r')(x_1)) &= u \\
x_1^h(r',u) &= \left(\frac{2750}{1.2}\right)^{\frac{1}{2}} = 47.87 \\
\Rightarrow \text{substitution effect} &= x_1^h(r',u) - x_1(r,I_1) \\
&= 47.87 - 50 = -2.13 \\
\text{income effect} &= x_1(r',I_2) - x_1^h(r',u) \\
&= 50 - 47.87 = 2.13
\end{aligned}
$$

Note that rise in the interest rate makes consumption in period 1 relatively more expensive than before. As a result the consumer will substitute away from period 1 consumption and save more. However, in this example, the income effect exactly cancels out this substitution effect.

14. Betty's utility function over leisure and consumption is

$$u(L,Y) = L^{\alpha}Y,$$

where $\alpha > 0$. Wage rate is $w = 5$ and the price of the composite consumption good is $p = 1$.

(a) Suppose Betty is currently working 8 hours. What must α be?

Solution: The utility maximization problem is given by

$$\max_{L,Y} L^{\alpha}Y \quad \text{s.t.} \quad wL + pY = 24w$$

Since $p = 1$, this simplifies to

$$\max_{L,Y} L^{\alpha}Y \quad \text{s.t.} \quad wL + Y = 24w$$

Recognizing that the utility function is a transformation of standard Cobb-Douglas utility function,

$$u(L,Y) = L^{\frac{\alpha}{\alpha+1}}Y^{\frac{1}{\alpha+1}},$$

we obtain the (Marshallian) demand as

$$
\begin{aligned}
L(w) &= \frac{\alpha}{\alpha+1}\left(\frac{24w}{w}\right) = \frac{24\alpha}{\alpha+1} \\
Y(w) &= \frac{1}{\alpha+1}\left(\frac{24w}{1}\right) = \frac{24w}{\alpha+1}.
\end{aligned}
$$

Since Betty is working 8 hours, we have

$$8 = 24 - L^* = 24 - \frac{24\alpha}{\alpha+1} \implies 24\alpha = 16(\alpha+1) \implies 8\alpha = 16 \implies \alpha = 2.$$

In the following, use the value of α found in part (a).

(b) Suppose Betty is given an additional \$40 lump-sum payment as a bonus. Find Betty's optimal consumption of leisure and the composite good. Will Betty work more or less than before?

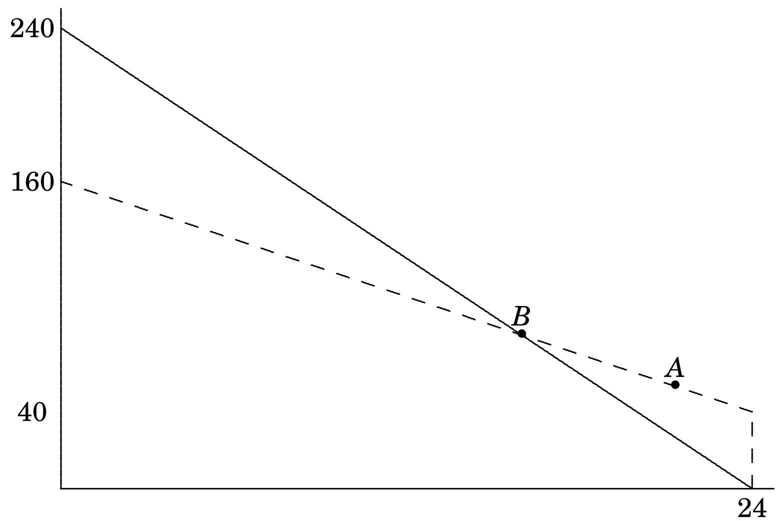
Solution: Receiving \$40 lump-sum bonus leads to the dashed-line budget set below.

$$\max_{L,Y} L^2Y \quad \text{s.t.} \quad wL + Y = 40 + 24w$$

$$\Longrightarrow \hat{L}(w) = \frac{2}{3}\left(\frac{40+24w}{w}\right) = \frac{2}{3}\left(\frac{40+24(5)}{5}\right) = \frac{2(160)}{3(5)} = 21.33$$

$$\hat{Y}(w) = \frac{1}{3}\left(\frac{40+24w}{1}\right) = \frac{160}{3} = 53.33,$$

which is point A on the graph below. Since $24 - 21.33 = 2.67 < 8$, Betty works fewer hours than before.



(c) Will Betty be happier if her wage rate doubles to $w' = 10$ instead of getting the \$40 lump-sum bonus?

Solution: Doubling the wage rate yields the solid-line budget set above. Betty's optimal consumption is

$$L(w') = \frac{24\alpha}{\alpha+1} = \frac{24(2)}{2+1} = 16$$

$$Y(w') = \frac{24w'}{\alpha+1} = \frac{24(10)}{2+1} = 80,$$

which is point B on the graph above.

To see which option Betty prefers,

$$U(A) = (21.33)^2(53.33) = 24,263.49$$

$$U(B) = (16)^2(80) = 20,480$$

So Betty prefers to receive the \$40 bonus.

15. A consumer's utility function over leisure and consumption is given by

$$u(L,Y) = LY.$$

Wage rate is w and the price of the composite consumption good is $p = 1$.

(a) Suppose $w = 10$. Find the optimal leisure - consumption combination.
Solution: The utility maximization problem is given by

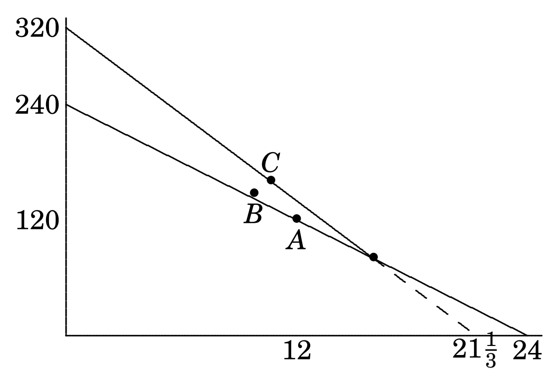
$$\max_{L,Y} LY \quad \text{s.t.} \quad wL + Y = 24w$$

Recognizing that this is a Cobb-Douglas utility function, we obtain the (Marshallian) demand as

$$L(w) = \frac{24w}{2w} = 12 \quad \text{and} \quad Y(w) = \frac{24w}{2} = 12w = 120,$$

represented by point A on the graph below.

(b) Suppose the overtime wage law is passed so that the firm must pay 1.5 times the normal wage for hours worked beyond the first 8 hours. Find the effect on the hours worked. Decompose the effect into substitution effect and income effect.
Solution:



Since the consumer currently works more than 8 hours, overtime law will have an effect. New consumption point will occur on the budget line given by

$$15L + Y = 21.33(15)$$

Again, using the short cut for Cobb-Douglas utility function yields

$$L(w') = \frac{21.33w'}{2w'} = 10.665 \quad \text{and} \quad Y(w') = \frac{24w'}{2} = 10.665(15) = 159.975,$$

which is point C on the graph. To find the expenditure minimizing way to achieve the original utility function at new wage $w' = 15$, we solve

$$\min_{L,Y} w'L + Y \quad \text{s.t.} \quad LY = (12)(120) = 1440$$

Setting MRS equal to the price ratio yields

$$\frac{Y}{L} = w' \quad \Rightarrow \quad Y = 15L.$$

Substituting this into the utility constraint yields point B (note that B is above the budget line passing through A):

$$L(15L) = 1440 \quad \Rightarrow \quad L' = \sqrt{\frac{1440}{15}} = 9.79$$
$$Y' = 15(9.79) = 146.85.$$

Therefore,

$$\begin{aligned} SE &= (9.79, 146.85) - (12, 120) = (-2.21, 26.85) \\ IE &= (10.67, 159.98) - (9.79, 146.85) = (0.88, 13.13) \\ TE &= (10.67, 159.98) - (12, 120) = (-1.33, 39.98). \end{aligned}$$

16. Consider a consumer that lives for two periods. The consumer's utility function over the consumption in the two periods is given by

$$u(x_1, x_2) = \min\{x_1, x_2\}.$$

Suppose she earns zero in period 1 but earns income I_2 in period 2. Assume that the consumer can borrow or lend freely at the gross interest rate $(1 + r)$.

(a) Set up the consumer's utility maximization problem.

Solution:

$$\max_{x_1, x_2} \min\{x_1, x_2\} \quad \text{s.t. } (1 + r)x_1 + x_2 = I_2.$$

(b) Letting $I_2 = 1000$ and $r = 0.1$, Find the optimal consumption choice and savings.

Solution: The solution x_1^* and x_2^* must satisfy $x_1^* = x_2^*$ and the budget equation $(1 + r)x_1^* + x_2^* = I_2$. Therefore,

$$x_1^* = x_2^* = \frac{I_2}{2 + r} = \frac{1000}{2.1} = 476.19$$

(c) Suppose the interest rate goes up to $r = 0.2$. How does the period 1 consumption change?

Solution:

$$x_1^* = x_2^* = \frac{I_2}{2 + r} = \frac{1000}{2.2} = 454.54$$

17. Consider again Question "Andy".

(a) Suppose that instead of the proposal given in that question, part (b), the government proposes the following "social security" plan. The government will take fraction t, where $0 < t < 1$, of the workers' income in their working years. When the workers retire, it will give each retiree ¥$(1+g)$, where $0 < g < 1$, for every ¥1 it collected from him/her. Assuming that Andy can still lend and borrow at interest rate r, derive Andy's lifetime budget constraint.

Solution: The budget constraints in each period is

$$\begin{aligned} (1) \quad c_1 &= (1-t)I_1 - S \\ (2) \quad c_2 &= (1+g)tI_1 + (1+r)S. \end{aligned}$$

Multiplying (1) by (1+r) yields:

$$\begin{aligned} (1+r)c_1 &= (1+r)(1-t)I_1 - (1+r)S \\ \Longleftrightarrow \quad (1+r)S &= (1+r)(1-t)I_1 - (1+r)c_1. \end{aligned}$$

Substituting this into (2) yields the budget constraint $(*)$

$$\begin{aligned} c_2 &= (1+g)tI_1 + (1+r)(1-t)I_1 - (1+r)c_1 \\ (1+r)c_1 + c_2 &= tI_1 + gtI_1 + (1+r)I_1 - tI_1 - rtI_1 \\ (*) \quad (1+r)c_1 + c_2 &= (1+r)I_1 + (g-r)tI_1. \end{aligned}$$

(b) Suppose as before the interest rate is $r = 0.2$ while the tax rate is $t = 0.10$ and the government's rate of return is $g = 0.2$. How will the proposed social security policy, if enacted, affect Andy's consumption and savings choice?

Solution: Note that in this problem, $r = 0.2 = g$. This implies that the budget constraint reduces to

$$(1+r)c_1 + c_2 = (1+r)I_1,$$

which is exactly the same as before the "social security" proposal was made. So, the solution will be same as Question "Andy" part (a), meaning the proposal will have no effect on Andy's behavior or utility.

(c) Explain your result above. In particular, what is the key fact in the problem that resulted in your answer (assuming, of course, that you obtained the correct answer)?

Solution: As seen above the key fact is that $g = r$. Since the return that the government offers is no different from what an individual can already get in the market, the policy has no effect.

(d) (optional) Can you say something general about when a government policy will be effective?

Solution: For the government policy to be effective, it must at least offer an opportunity that the market does not already provide. In this example, it means we need $g \neq r$. Andy will be better off (relative to no government policy) if $g > r$ and worse off if $g < r$.

18. A consumer who lives for two periods has a standard Cobb-Douglas utility function:

$$u(c_1, c_2) = c_1^{\alpha} c_2^{\beta},$$

where c_t = consumption in period t and $\alpha + \beta = 1$. Her income in period one is $I_1 = 500$ and $I_2 = 400$, and she can lend or borrow at interest rate $r = 0.2$.

(a) Find the optimal consumption demand.

Solution: The budget equation is

$$\begin{aligned}(1+r)c_1 + c_2 &= (1+r)I_1 + I_2 \\ (1.2)c_1 + c_2 &= (1.2)500 + 400 = 1000.\end{aligned}$$

So the demand functions are:

$$\begin{aligned}c_1 &= \alpha\left(\frac{(1+r)I_1 + I_2}{1+r}\right) = \alpha\left(\frac{1000}{1.2}\right) \approx \alpha(833.33) \\ c_2 &= \beta\left(\frac{(1+r)I_1 + I_2}{1}\right) = \beta(1000).\end{aligned}$$

(b) What values of α, if any, makes the consumer a borrower? Interpret this result.

Solution: For the consumer to be a borrower, we need

$$\alpha(833.33) = c_1 > I_1 = 500 \quad \Rightarrow \quad \alpha > \frac{500}{833.33} \approx 0.60.$$

The exponents α and β represent the relative contributions of consumption in the two periods to the consumer's overall utility. Higher α means period 1 consumption is relatively more important, which makes the consumer more likely to borrow from the future to consume more in period 1.

(c) Suppose now that $\alpha = \frac{4}{7}$ but that r is no longer 0.2. What values of r, if any, makes the consumer a borrower? Interpret this result.

Solution: For the consumer to be a borrower, we need

$$\begin{aligned}\alpha\left(\frac{(1+r)I_1 + I_2}{1+r}\right) &= \alpha\left(I_1 + \frac{I_2}{1+r}\right) = c_1 > I_1 = 500 \\ \Rightarrow \quad \frac{400}{1+r} &> \frac{7}{4}(500) - 500 \quad \Rightarrow \quad 400 > 375(1+r) \\ \Rightarrow \quad r &< \frac{25}{375} = 0.0667\end{aligned}$$

The interest rate r is the cost of borrowing. So, lower interest rate makes borrowing more likely.

19. Consider the utility function:

$$u(x_1, x_2) = x_1 + x_2.$$

Find the corresponding Hicksian demand function.

Solution: Since the indifference curves are linear, the solution will not be unique. In fact, at interior solutions,

$$MRS = 1 = \frac{p_1}{p_2}.$$

This implies that the interior solution occurs only when $p_1 = p_2$ and that any bundle on the indifference curve is optimal. When $p_1 \neq p_2$, then the solution is on the boundary. For example, if $p_1 < p_2$ then the expenditure is minimized by consuming only good 1. To summarize:

$$x^h(p,u) = \begin{cases} (u,0) & \text{if } p_1 < p_2 \\ \{(x_1, u - x_1) : 0 \leq x_1 \leq u\} & \text{if } p_1 = p_2 \\ (0,u) & \text{if } p_1 > p_2. \end{cases}$$

20. Consider a utility-maximizer in a two goods world. Is it possible that every good is a Giffen good? What if there are more than 2 goods? Explain.

Solution: Suppose there are $N \geq 1$ goods. We know that the optimal consumption must occur on the budget line. I.e., $x(p,I)$ must satisfy:

$$\begin{aligned} & p_1 x_1(p,I) + p_2 x_2(p,I) + \ldots + p_N x_N(p,I) &=& \ I \\ \Rightarrow \quad & \frac{\partial}{\partial I}\left[p_1 x_1(p,I) + p_2 x_2(p,I) + \ldots + p_N x_N(p,I)\right] &=& \ \frac{\partial}{\partial I}[I] \\ \Rightarrow \quad & p_1 \frac{\partial x_1(p,I)}{\partial I} + p_2 \frac{\partial x_2(p,I)}{\partial I} + \ldots + p_N \frac{\partial x_N(p,I)}{\partial I} &=& \ 1 \quad (*) \end{aligned}$$

If all the goods are inferior then the left hand side of the equation $(*)$ will be negative so that the equation cannot be satisfied. Therefore, not every good can be inferior. Since Giffen good must be inferior, this means that not every good can be Giffen.

21. Let h_ℓ denote the Hicksian demand for good ℓ. Say that good ℓ and good k are complements if $\frac{\partial h_\ell}{\partial p_k} \leq 0$.

    (a) Suppose there are only two goods in the economy. Can the two goods be complements?

    Solution: Suppose the two goods are strict complements so that $\frac{\partial h_\ell}{\partial p_k} < 0$. That is, when p_k increases h_ℓ decreases. But then h_k would have to increase to keep consumer's utility the same. This is not possible since Hicksian demand always satisfies the law of demand. Thus, only way in which two goods can be complements is if, in fact, $\frac{\partial h_\ell}{\partial p_k} = 0$.

(b) Now, suppose there are more than two goods in the economy. Can they all be complements?

Solution: By similar reasoning as above, they cannot all be strict complements.

22. For each of the three utility functions below, find the substitution effect, the income effect, and the total effect that result when prices change from $p = (2,1)$ (i.e., $p_1 = 2$ and $p_2 = 1$) to $p' = (2,4)$ (i.e., $p'_1 = 2$ and $p'_2 = 4$). Assume the consumer has income $I = 20$.

(a) Before doing any calculation, make an educated guess about the relative magnitudes of the substitution effects for the three utility functions below. Be sure to explain the reasoning behind your guesses.

Solution: Since the utility function in (b) is that of perfect substitutes, changes in the relative prices should have an extreme effect on the consumption. In contrast, the utility function in (d) is that of perfect complements with zero substitutability, while the Cobb-Douglas utility function in (c) has some degree of substitutability. Therefore, a reasonable guess would be $|SE_i^b| > |SE_i^c| > |SE_i^d| = 0$ for both goods.

(b) $u(x_1,x_2) = x_1 + x_2$.

Solution: At price $p = (2,1)$,

$$\frac{MU_1}{MU_2} = 1 < \frac{2}{1} = \frac{p_1}{p_2},$$

So, the consumer will only consume good 2: $x(p,w) = (0, \frac{I}{p_2}) = (0,20)$, and the consumer's utility is $u(0,20) = 20$. If the price of good 2 rises to 4, then now

$$\frac{MU_1}{MU_2} = 1 > \frac{2}{4} = \frac{p'_1}{p'_2},$$

So, the consumer will switch to consuming only good 1. In order to maintain utility $u = 20$, this means $x^h(p',u) = (20,0)$. Finally, since the consumer's income is unchanged at $I = 20$, the consumer will consumer $x(p',I) = (\frac{I}{p'_1}, 0) = (10,0)$. Therefore, we have

$$\begin{array}{rcccc} TE & = & SE & + & IE \\ x(p',I) - x(p,I) & = & x^h(p',u) - x(p,I) & + & x(p',I) - x^h(p',u) \\ (10,0) - (0,20) & = & (20,0) - (0,20) & + & (10,0) - (20,0) \\ (10,-20) & = & (20,-20) & + & (-10,0). \end{array}$$

(c) $u(x_1,x_2) = x_1 x_2$.

Solution: From earlier examples, we know that

$$\begin{aligned} x(p,I) &= \left(\frac{I}{2p_1}, \frac{I}{2p_2}\right) = \left(\frac{20}{2(2)}, \frac{20}{2(1)}\right) = (5,10), \quad \text{and} \\ x(p',I) &= \left(\frac{20}{2(2)}, \frac{20}{2(4)}\right) = (5,2.5) \end{aligned}$$

and the consumer's original utility is $u(5,10) = 50$. After some calculation, the Hicksian demand can found to be

$$\begin{aligned} x^h(p',u) &= \left(\left(\frac{p_2' u}{p_1'}\right)^{\frac{1}{2}}, \left(\frac{p_1' u}{p_2'}\right)^{\frac{1}{2}}\right) = \left(\left(\frac{4(50)}{2}\right)^{\frac{1}{2}}, \left(\frac{2(50)}{4}\right)^{\frac{1}{2}}\right) \\ &= \left((100)^{\frac{1}{2}}, (25)^{\frac{1}{2}}\right) = (10,5). \end{aligned}$$

So,

$$\begin{array}{ccccc} TE & = & SE & + & IE \\ x(p',I) - x(p,I) & = & x^h(p',u) - x(p,I) & + & x(p',I) - x^h(p',u) \\ (5,2.5) - (5,10) & = & (10,5) - (5,10) & + & (5,2.5) - (10,5) \\ (0,-7.5) & = & (5,-5) & + & (-5,-2.5). \end{array}$$

(d) $u(x_1,x_2) = \min\{x_1,x_2\}$.

Solution: Marshallian demand occurs where the "kink" equation $x_1 = x_2$ and the budget equation $p_1x_1 + p_2x_2 = I$ intersect. Therefore,

$$\begin{aligned} x(p,I) &= \left(\frac{I}{p_1+p_2}, \frac{I}{p_1+p_2}\right) = \left(\frac{20}{2+1}, \frac{20}{2+1}\right) = (6.67, 6.67), \\ x(p',I) &= \left(\frac{20}{2+4}, \frac{20}{2+4}\right) = (3.33, 3.33), \end{aligned}$$

and the consumer's original utility is $u(6.67, 6.67) = 6.67$. The Hicksian demand occurs where the "kink" equation $x_1 = x_2$ intersects the utility constraint $\min\{x_1,x_2\} = 6.67$. Thus,

$$x^h(p',u) = (6.67, 6.67)$$

So,

$$\begin{array}{ccccc} TE & = & SE & + & IE \\ x(p',I) - x(p,I) & = & x^h(p',u) - x(p,I) & + & x(p',I) - x^h(p',u) \\ (3.33,3.33) - (6.67,6.67) & = & (6.67,6.67) - (6.67,6.67) & + & (3.33,3.33) - (6.67,6.67) \\ (-3.34,-3.34) & = & (0,0) & + & (-3.34,-3.34). \end{array}$$

(e) Rank the substitution effects found above by their magnitude (do this for each good). To what extent does it conform to your guess?

Solution: For substitution effect, we have

$$|SE_i^b| > |SE_i^c| > |SE_i^d| = 0$$

for each good i as we had expected. (Can you think of a situation where this would not have been the outcome?)

23. Consider again the Problem set 2, Question 1, where income was $I = 10$, and the price of good 1 changed from $p_1 = 2$ to $p_1' = 4$ while the price of good 2 stayed the same at $p_2 = 1$.

(a) Find the total effect of the price change on good 1.

Solution: The total effect is $TE_1 = x_1(p'_1, p_2, I) - x_1(p_1, p_2, I) = 0 - 3 = -3$.

(b) Find the substitution effect on good 1.

Solution: The original utility was $u(3,4) = 3 + 2\sqrt{4} = 7$. From Question 2, we have $x_2^h = 16$. But trying to solve

$$u(x_1^h, 16) \;=\; x_1^h + 2\sqrt{16} \;=\; 7$$

leads to $x_1^h < 0$. So, we have a boundary solution here as well. We have $x_1^h = 0$ and

$$0 + 2\left(x_2^h\right)^{\frac{1}{2}} = 7 \quad \Rightarrow \quad x_2^h = 12.25.$$

So, the substitution effect is $SE_1 = x_1^h(p'_1, p_2, u^o) - x_1(p_1, p_2, I) = 0 - 3 = -3$.

(c) Find the income effect on good 1.

Solution: The income effect is $IE_1 = x_1(p'_1, p_2, I) - x_1^h(p'_1, p_2, u^o) = 0 - 0 = 0$.

24. Consider a utility-maximizer in a two goods world. Is it possible that every good is an inferior good? What if there are more than 2 goods? Explain.

Solution: Suppose there are $N \geq 1$ goods. We know that the optimal consumption must occur on the budget line. I.e., $x(p, I)$ must satisfy:

$$\begin{aligned}
p_1 x_1(p,I) + p_2 x_2(p,I) + \ldots + p_N x_N(p,I) &= I \\
\Rightarrow \quad \frac{\partial}{\partial I}\left[p_1 x_1(p,I) + p_2 x_2(p,I) + \ldots + p_N x_N(p,I)\right] &= \frac{\partial}{\partial I}[I] \\
\Rightarrow \quad p_1 \frac{\partial x_1(p,I)}{\partial I} + p_2 \frac{\partial x_2(p,I)}{\partial I} + \ldots + p_N \frac{\partial x_N(p,I)}{\partial I} &= 1 \quad (*)
\end{aligned}$$

If all the goods are inferior then the left hand side of the equation $(*)$ will be negative so that the equation cannot be satisfied. Therefore, not every good can be inferior.