Intermediate Microeconomics

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Midterm examination: suggested solutions

1. [20] Consider a leisure-consumption model in which an individual is deciding her daily consumption of leisure and the composite consumption good. Her utility is $u(L,Y) = a\ln L + Y$, where L and Y denote the amount of leisure and the composite good, respectively. The individual has 24 hours of time in total. The hourly wage is w, and the price of the consumption good is p.

   (a) [7] Find the individual's (Marshallian) demand for leisure and the consumption good.

   Solution: Solving the first order condition for utility maximization yields,

   $$MRS = \frac{\frac{\partial u}{\partial L}}{\frac{\partial u}{\partial Y}} = \frac{a}{L} = \frac{w}{p} \implies L^* = \frac{ap}{w} \text{ and } Y^* = \frac{24w - wL^*}{p} = \frac{24w - ap}{p}.$$

   (b) [6] For what values of a, if any, will the individual not work? Explain this result.

   Solution: The individual will not work if $L^* = \frac{ap}{w} \geq 24 \iff a \geq \frac{24w}{p}$. Note that as a increases, the individual's value for leisure relative to the consumption good increases; therefore, her incentive to work decreases, holding everything else constant. Indeed, if $a \geq \frac{24w}{p}$ then her MRS_{LY} at 24 hours of leisure is $\frac{a}{L} \geq \frac{24w}{24p} = \frac{w}{p}$. If the weak inequality holds with equality then that means 24 hours of leisure (zero hours of work) is optimal. If it holds with strict inequality, then it means that the (relative) market price of leisure is cheaper than her (relative) valuation for leisure. Thus, she in fact wants to buy more leisure, but she cannot because she's already consuming the maximum amount of leisure possible.

   (c) [7] Suppose the hourly wage increases from $w = 2$ to $w' = 4$ while the price of the composite good remains fixed at $p = 1$. Find the substitution, the income, and the total effects on leisure.

   Solution: For this utility function, the Marshallian demand and the Hicksian demands for leisure is determined solely by the "MRS = price ratio" condition. Therefore, they are the same. Thus, assuming interior solution and letting u^0 denote the utility at the original Marshallian demand, $L(w,I)$ and $Y(w,I)$, we have:

   $$\begin{aligned} TE_L &= x_L(w',I) - x_L(w,I) = \frac{a}{4} - \frac{a}{2} = -\frac{a}{4} \\ SE_L &= h_L(w',u^0) - x_L(w,I) = \frac{a}{4} - \frac{a}{2} = -\frac{a}{4} \\ \implies IE_L &= TE_L - IE_L = 0. \end{aligned}$$

2. [20] Consider a two-period consumption model in which an individual's lifetime utility is $u(c_1, c_2) = \min\ \{c_1, \frac{c_2}{a}\}$, where c_t is the consumption in period t and $a > 0$. The individual receives income I_1 in period 1 and I_2 in period 2. The price levels in both periods are one ($p_1 = p_2 = 1$).

(a) [7] Suppose the individual can borrow or save freely at a bank at a net interest rate $r > 0$. Find the values of a for which the individual will be a saver and not a borrower. Explain this result.

Solution: For a Leontieff utility, we substitute "corner equation", $c_2 = ac_1$, into the budget constraint:

$$(1+r)c_1 + ac_1 = (1+r)I_1 + I_2 \implies c_1^* = \frac{(1+r)I_1 + I_2}{1+r+a}.$$

The individual will be a saver if

$$I_1 > c_1^* \iff I_1 > \frac{(1+r)I_1 + I_2}{1+r+a} \iff (1+r+a)I_1 > (1+r)I_1 + I_2 \iff a > \frac{I_2}{I_1}.$$

As a increases, the consumption in period 2 becomes more important to the individual, and her incentive to save increases. Note that if her income in period 2 relative to period 1, $\frac{I_2}{I_1}$, increases then the need for saving decreases. Thus, the relative importance of period 2 consumption that is need to turn her into a saver also increases.

For parts (b) and (c) below, assume $a = 1$ and $I_1 > I_2$.

(b) [6] Now, suppose that there is no bank and that the only way to save in this economy is through a government program that allows individuals to save up to 25% of their period 1 income at a net interest rate $g > r$. When will the cap on the amount that can be saved be binding?

Solution: Ignoring the cap, the budget constraint is the same as part (a), except r is replaced by g. Thus, the optimal consumption is now

$$c_1^G = \frac{(1+g)I_1 + I_2}{1+g+a} = \frac{(1+g)I_1 + I_2}{2+g}, \text{ provided that this is } \geq 0.75I_1$$

The cap is binding if $c_1^G = \frac{(1+g)I_1+I_2}{2+g} < \frac{3I_1}{4}$. That is,

$$4(1+g)I_1 + 4I_2 < 3(2+g)I_1 \iff 4I_2 < 2I_1 - gI_1 \iff g < \frac{2I_1 - 4I_2}{I_1}.$$

(c) [7] When will the situation in part (b) be better than the situation in part (a) for the individual?

Solution: For Leontieff utility, a higher period 1 consumption means higher lifetime utility. We have

$$\frac{d}{dx}\left(\frac{(1+x)I_1 + I_2}{2+x}\right) = \frac{I_1(2+x) - ((1+x)I_1 + I_2)}{(2+x)^2} = \frac{I_1 - I_2}{(2+x)^2} > 0.$$

So, if the cap is not binding, the government program is better since $c_1^* < c_1^G$. If the cap is binding, the government program will be better if

$$c_1^* = \frac{(1+r)I_1 + I_2}{2+r} < \frac{3I_1}{4} = c_1^g \iff r < \frac{2I_1 - 4I_2}{I_1} \text{ by replacing } g \text{ with } r \text{ in (b).}$$

3. [20] Consider an individual whose utility function over wealth is $u(w) = \ln w$.

(a) [4] What is the individual's Arrow-Pratt measure of absolute risk aversion and the Arrow-Pratt measure of relative risk aversion? How do they change as her wealth changes?

Solution: Since $u'(w) = \frac{1}{w}$ and $u''(w) = -\frac{1}{w^2}$, we have

$$R_A(w) = -\frac{u''(w)}{u'(w)} = -\frac{-\frac{1}{w^2}}{\frac{1}{w}} = \frac{1}{w} \quad \text{(DARA)}$$

$$R_R(w) = -\frac{wu''(w)}{u'(w)} = -\frac{-w\frac{1}{w^2}}{\frac{1}{w}} = \frac{\frac{1}{w}}{\frac{1}{w}} = 1 \quad \text{(CRRA)}.$$

(b) [6] Suppose the individual's current wealth is W, and she faces a risk in which she could gain 25% of her wealth with probability $\frac{1}{2}$ or lose 60% of her wealth with probability $\frac{1}{2}$. What is the maximum amount of money that she is willing to pay to avoid this risk?

Solution: Letting M be the maximum willingness to pay, we have

$$\ln(W - M) \geq \frac{1}{2}\ln(1.25W) + \frac{1}{2}\ln(0.4W) = \ln\left((1.25W)^{\frac{1}{2}}(0.4W)^{\frac{1}{2}}\right) = \ln\left(0.5^{\frac{1}{2}}W\right)$$

$$\implies W - M \geq 0.5^{\frac{1}{2}}W \implies M \leq (1-\sqrt{.5})W = (1-0.707)W = 0.293W.$$

(c) [4] Suppose the individual did not pay and took the risk and that after the gain or the loss occurs she faces the exact same risk in part (b) again. What is the maximum amount she will pay to avoid the risk now?

Solution: From part (b), we know that her maximum willingness to pay will be $0.293(1.25W) = 0.366W$ if she had gained and $0.293(0.4W) = 0.117W$ if she had lost.

(d) [6] Suppose the individual faced the exact same risk in part (b) n times already and is facing it yet again. What is the maximum amount she will pay to avoid the risk now? How about the maximum she is willing to pay as a share of her wealth? Explain this result in relation to part (a).

Solution: Let m, where $0 \leq m \leq n$, be the number of times the individual gained. Then her current wealth is $W_n = W(1.25)^m(0.4)^{n-m}$. So the maximum amount she is willing to pay is $0.293W_n = 0.293(1.25)^m(0.4)^{n-m}W$. Note that no matter what her wealth is currently, the maximum she is willing to pay in relative terms (as a share of wealth) is constant 0.293. This is because she has a CRRA utility, which means her attitude toward a risk that are given in relative terms is independent of her wealth level.

4. [20] Consider a strictly risk averse individual with utility function over wealth $u(w)$. The individual has an initial wealth W RMB and faces a risk of losing L RMB with probability $\alpha > \frac{1}{2}$, or gaining G RMB with probability $1-\alpha$, where $L > 0$, $G > 0$, and $L + G < W$.

   (a) [8] Suppose an insurance is available at price p per unit, where each unit of insurance pays 1 RMB if a loss occurs and nothing otherwise. Assuming that the price is fair, find the optimal amount of insurance the individual should buy under two cases: first when the maximum she can buy is the amount of the loss (L), and second when the maximum is the amount of her wealth (W).

   Solution: We have

$$\begin{aligned} U(x) &= \alpha u\big(W - L + (1-p)x\big) + (1-\alpha)u\big(W + G - px\big) \\ U'(x) &= \alpha u'\big(W - L + (1-p)x\big)(1-p) - (1-\alpha)u'\big(W + G - px\big)p. \\ U''(x) &= \alpha u''\big(W - L + (1-p)x\big)(1-p)^2 + (1-\alpha)u''\big(W + G - px\big)p^2 < 0. \end{aligned}$$

   The first order condition for interior solution with fair price $p = \alpha$ yields,

$$\begin{aligned} \alpha u'\big(W - L + (1-\alpha)x\big)(1-\alpha) &= (1-\alpha)u'\big(W + G - \alpha x\big)\alpha \\ u'\big(W - L + (1-\alpha)x\big) &= u'\big(W + G - \alpha x\big) \\ \implies W - L + (1-p)x &= W + G - px \text{ since } U' \text{ is decreasing} \\ \implies x &\geq G + L. \end{aligned}$$

   Thus, $x^* = L$ if the maximum amount she can buy is L (since $U' > 0$), and $x^* = G + L$ if the maximum she can buy is W.

   (b) [2] Now, assume that each unit of "insurance" pays the individual 1 RMB if the loss occurs and collects 1 RMB if the gain occurs. Find the fair price for this insurance.

   Solution: The zero profit condition for the insurance company is now

$$px - \alpha x + (1-\alpha)x = px - \alpha x + x - \alpha x = 0 \implies p = 2\alpha - 1.$$

   (c) [10] Continuing part (b), assuming that the price is fair and find the optimal amount of insurance the individual should buy.

   Solution: Assuming interior solution, we have

$$\begin{aligned} U(x) &= \alpha u\big(W - L + (1-p)x\big) + (1-\alpha)u\big(W + G - (1+p)x\big) \\ U'(x) &= \alpha u'\big(W - L + (1-p)x\big)(1-p) - (1-\alpha)u'\big(W + G - (1+p)x\big)(1+p). \\ &= \alpha u'\big(W - L + (1-p)x\big)(1 - 2\alpha + 1) - (1-\alpha)u'\big(W + G - (1+p)x\big)(1 + 2\alpha - 1) \\ &= 2\alpha(1 - 1\alpha)u'\big(W - L + (1-p)x\big) - 2\alpha(1-\alpha)u'\big(W + G - (1+p)x\big) \end{aligned}$$

   The first order condition for interior solution yields,

$$\begin{aligned} & u'\big(W - L + (1-p)x\big) = u'\big(W + G - (1+p)x\big) \\ & \implies W - L + (1-p)x = W + G - (1+p)x \\ & \quad x - px + (x + px) = G + L \implies x^* = \frac{G+L}{2} \text{ (SOC is satisfied as in part (a)).} \end{aligned}$$

5. [20] A firm's production function is given by $f(L,K) = L^a K^{\frac{1}{3}}$, where $a > 0$. Let w be the price of labor and r be the price of capital.

   (a) [6] Find the firm's short-run cost function, first when it is the amount labor that is fixed, and second when it is the amount of capital that is fixed. To keep the calculations simple, assume that labor is fixed at $\bar{L}^{\frac{1}{a}}$ and capital is fixed at $\bar{K}^3$, respectively, for the two cases.

   Solution: Suppose labor is fixed at $\bar{L}^{\frac{1}{a}}$ in the SR. Using $f(\bar{L},K) = q$, we have

$$\left(\bar{L}^{\frac{1}{a}}\right)^a K^{\frac{1}{3}} = q \implies K(q) = \left(\frac{q}{\bar{L}}\right)^3 \implies c(q) = w\bar{L}^{\frac{1}{a}} + rK(q) = w\bar{L}^{\frac{1}{a}} + r\left(\frac{q}{\bar{L}}\right)^3.$$

   Suppose instead capital is fixed at $\bar{K}^3$ in the SR. Using $f(L,\bar{K}) = q$, we obtain

$$L^a(\bar{K}^3)^{\frac{1}{3}} = q \implies L(q) = \left(\frac{q}{\bar{K}}\right)^{\frac{1}{a}} \implies C(q) = wL(q) + r\bar{K}^3 = r\bar{K}^3 + w\left(\frac{q}{\bar{K}}\right)^{\frac{1}{a}}.$$

   (b) [7] Continuing part (a), show that this firm's short-run profit maximization problem always has a solution if it is the amount labor that is fixed but may not have a solution if it is the amount of capital that is fixed.

   Solution: When labor is fixed, $p = MC$ condition yields

$$p = \frac{3rq^2}{\bar{L}^3} \implies q(p) = \left(\frac{p\bar{L}^3}{3r}\right)^{\frac{1}{2}} = \left(\frac{p}{3r}\right)^{\frac{1}{2}} \bar{L}^{\frac{3}{2}}.$$

   Since MC is always increasing, and min $AVC = \min \frac{rq^2}{\bar{L}^3} = 0$, this is indeed the solution to the firm's profit maximization problem. In contrast, when capital is fixed, we have

$$MC = \frac{w}{a\bar{K}^{\frac{1}{a}}} q^{\frac{1}{a}-1} = \frac{w}{a\bar{K}^{\frac{1}{a}}} q^{\frac{1-a}{a}} \implies \frac{dMC}{dq} = \frac{(1-a)w}{a^2\bar{K}^{\frac{1}{a}}} q^{\frac{1-a}{a}}.$$

   That is, MC is always decreasing if $a > 1$. Thus, the firm will want to always produce a little more, and the profit maximization problem has no solution (the firm wants to produce an infinite amount).

   (c) [7] Assuming that $a = \frac{1}{3}$, find the firm's long-run cost function.

   Solution: Since $f(L,K) = L^{\frac{1}{3}} K^{\frac{1}{3}}$, we have

$$MRTS = \frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}} = \frac{\frac{1}{3}L^{\frac{-2}{3}} K^{\frac{1}{3}}}{\frac{1}{3}L^{\frac{1}{3}} K^{\frac{-2}{3}}} = \frac{K}{L} = \frac{w}{r} \implies K = \frac{wL}{r}.$$

   Substituting this into the output constraint $f(L,K) = q$ yields,

$$L^{\frac{1}{3}}\left(\frac{wL}{r}\right)^{\frac{1}{3}} = q \implies L(q) = \left(\frac{r^{\frac{1}{3}}}{w^{\frac{1}{3}}} q\right)^{\frac{3}{2}} = \left(\frac{r}{w}\right)^{\frac{1}{2}} q^{\frac{3}{2}} \implies K(q) = \frac{w}{r}\left(\left(\frac{r}{w}\right)^{\frac{1}{2}} q^{\frac{3}{2}}\right) = \left(\frac{w}{r}\right)^{\frac{1}{2}} q^{\frac{3}{2}}$$

$$\implies c(q) = wL(q) + rK(q) = w\left(\frac{r}{w}\right)^{\frac{1}{2}} q^{\frac{3}{2}} + r\left(\frac{w}{r}\right)^{\frac{1}{2}} q^{\frac{3}{2}} = 2(wr)^{\frac{1}{2}} q^{\frac{3}{2}}.$$