

Advanced Microeconomics I

Fall 2026 - M. Pak and B. Xu

Problem Set 1: Suggested Solutions

1. Let $u(x) = ax_1 + bx_2^{\frac{1}{2}}$, where $a, b \in \mathbb{R}$, be a utility function representing a preference relation $\succsim$ on the consumption set $\mathbb{R}_+^2$.

- (a) Find the conditions on a and b under which $u(\cdot)$ is locally non-satiated, weakly monotone, and strongly monotone, respectively. Which of the three properties is the most restrictive? The least restrictive?

Solution: If $a \leq 0$ and $b \leq 0$, then the utility is weakly decreasing in both x_1 and x_2 , so there is no bundle that is better than $(0, 0)$ in the consumption set. Thus, $\succsim$ is not locally non-satiated. But if $a > 0$, then for any $x = (x_1, x_2)$ and any neighborhood of radius $\varepsilon > 0$, we can find small enough γ such that $y = (x_1 + \gamma, x_2) \succ (x_1, x_2)$ and y is in that neighborhood. Similarly, if $b > 0$, then for any $x = (x_1, x_2)$ and any neighborhood of radius $\varepsilon > 0$, we can find small enough γ such that $y = (x_1, x_2 + \gamma) \succ (x_1, x_2)$ and y is in that neighborhood. Thus, $\succsim$ is locally non-satiated on $\mathbb{R}_+^2$ if and only if $a > 0$ or $b > 0$ (note that “or” means at least one of the two inequalities has to hold, so for example $a > 0$ and $b < 0$ is allowed). For (weak) monotonicity, we need $a \geq 0$ and $b \geq 0$ and at least one of the inequality to hold strictly. For strong monotonicity, we need $a > 0$ and $b > 0$. Thus, strong monotonicity is the most restrictive and local non-satiation is the least restrictive.

Solution to the utility maximization problem, denoted $x(p, w)$, is called the Marshallian demand and gives the individual’s optimal consumption bundle as a function of prices and wealth. The individual’s utility level at the optimal consumption, $v(p, w) = u(x(p, w))$, is called the indirect utility function. For the remainder of the question, assume that $a = 10$, $b = 2$, and $p \gg 0$.

- (b) Assuming that the solution to the utility maximization problem is interior (that is, $x(p, w) \gg 0$), find the individual’s Marshallian demand and the indirect utility. Verify that the indirect utility function is a linear function of w in this example.

Solution: Let

$$\mathcal{L}(x_1, x_2, \lambda) = 10x_1 + 2x_2^{\frac{1}{2}} + \lambda(w - p_1x_1 - p_2x_2).$$

The first order conditions are:

$$\begin{aligned} (1) \quad & \frac{\partial \mathcal{L}}{\partial x_1} = 10 - \lambda p_1 \leq 0 \quad (= 0 \text{ if } x_1 > 0) \\ (2) \quad & \frac{\partial \mathcal{L}}{\partial x_2} = x_2^{-\frac{1}{2}} - \lambda p_2 \leq 0 \quad (= 0 \text{ if } x_2 > 0) \\ (3) \quad & \frac{\partial \mathcal{L}}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0. \end{aligned}$$

Interior solution means (1) and (2) hold with equality. Thus, dividing (1) by (2) yields

$$10x_2^{\frac{1}{2}} = \frac{p_1}{p_2} \implies x_2^* = \left(\frac{p_1}{10p_2} \right)^2.$$

Substitute this into the budget equation to obtain:

$$x_1^* = \frac{w - p_2 \left(\frac{p_1}{10p_2} \right)^2}{p_1} = \frac{w}{p_1} - \frac{p_1}{100p_2}.$$

Organizing the solution into a row vector for convenience yields,

$$x(p, w) = (x_1(p, w), x_2(p, w)) = \left(\frac{w}{p_1} - \frac{p_1}{100p_2}, \left(\frac{p_1}{10p_2} \right)^2 \right).$$

Put thing in the utility function yields the indirect utility function:

$$v(p, w) = 10 \left(\frac{w}{p_1} - \frac{p_1}{100p_2} \right) + 2 \left(\frac{p_1}{10p_2} \right) = \frac{p_1}{10p_2} + \left(\frac{10}{p_1} \right) w,$$

which is linear in w .

- (c) Under what conditions will the Marshallian demand occur on the boundary (that is, $x_1(p, w) = 0$ or $x_2(p, w) = 0$)? What is the individual's Marshallian demand and the indirect utility in this case?

Solution: In part (b), we see from the expression for x_1^* in part (b) that an interior solution is obtained only if $w > \frac{p_1^2}{100p_2}$. Thus, when $w \leq \frac{p_1^2}{100p_2}$, the non-negativity constraint will force $x_1^* = 0$, and then the budget equation yields $x_2^* = \frac{w}{p_2}$. Thus, the Marshallian demand is

$$x(p, w) = (x_1(p, w), x_2(p, w)) = \left(0, \frac{w}{p_2} \right) \quad \text{if } w \leq \frac{p_1^2}{100p_2}.$$

The corresponding indirect utility function is

$$v(p, w) = 2 \left(\frac{w}{p_2} \right)^{\frac{1}{2}}.$$

2. Mas-Colell, Whinston, and Green (MWG) 3.C.6. For part (b), you may restrict attention to $x \gg 0$. In addition, it may be easier to work with the logarithmic transformation of the utility in part (b), $u(x) = \frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho}$, assuming that $\alpha_1 + \alpha_2 = 1$ (but explain why you are justified in doing this).

Solution: Let $u(x)$ be a CES utility function

$$u(x) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}$$

- (a) As $\rho \rightarrow 1$, the utility function converges to $\alpha_1 x_1 + \alpha_2 x_2$, which is linear.
- (b) Recall that a strictly increasing transformation of a utility function does not change the underlying preference. So we can assume $\alpha_1 + \alpha_2 = 1$ without loss of generality and can also use the natural log of the CES utility function:

$$u(x) = \frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho}$$

Since both the numerator and the denominator converges to zero as $\rho \rightarrow 0$, we can use L'Hopital's rule:

$$\begin{aligned} \lim_{\rho \rightarrow 0} u(x) &= \lim_{\rho \rightarrow 0} \frac{\frac{\partial \ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\partial \rho}}{\frac{\partial \rho}{\partial \rho}} \\ &= \lim_{\rho \rightarrow 0} \frac{\alpha_1 x_1^\rho \ln x_1 + \alpha_2 x_2^\rho \ln x_2}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \\ &= \alpha_1 \ln x_1 + \alpha_2 \ln x_2. \end{aligned}$$

This is a natural log transformation of a standard Cobb-Douglas utility function

$$u(x) = x_1^{\alpha_1} x_2^{\alpha_2}.$$

- (c) We need to show that

$$\lim_{\rho \rightarrow -\infty} u(x) = \lim_{\rho \rightarrow -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} = \min\{x_1, x_2\}.$$

Without loss of generality, we can let $x_1 = \min\{x_1, x_2\}$. Consider any $\rho < 0$. Since $x_1 \leq x_2$, we have

$$\begin{aligned} \alpha_1 x_1^\rho &\leq \alpha_1 x_1^\rho + \alpha_2 x_2^\rho \leq \alpha_1 x_1^\rho + \alpha_2 x_1^\rho = (\alpha_1 + \alpha_2) x_1^\rho \\ \alpha_1^{\frac{1}{\rho}} x_1 &\geq (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} \geq (\alpha_1 + \alpha_2)^{\frac{1}{\rho}} x_1 \end{aligned}$$

Letting $\rho \rightarrow -\infty$, we obtain,

$$x_1 \geq \lim_{\rho \rightarrow -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} \geq x_1,$$

as required.

3. MWG 3.D.5, parts (a) and (d). (You may assume that the solution will be interior)

Solution: Let $u(x) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$.

- (a) To find the demand function, we may use the transformation $\tilde{u}(x) = (u(x))^\rho = x_1^\rho + x_2^\rho$.

The first order condition for an interior solution is:

$$\begin{aligned} \frac{\frac{\partial \tilde{u}}{\partial x_1}}{\frac{\partial \tilde{u}}{\partial x_2}} &= \frac{\rho x_1^{\rho-1}}{\rho x_2^{\rho-1}} = \frac{p_1}{p_2} \Rightarrow x_1 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} x_2 \\ \Rightarrow p_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} x_2 + p_2 x_2 &= w \\ \Rightarrow \left(\frac{p_1^\rho}{p_2}\right)^{\frac{1}{\rho-1}} x_2 + \left(\frac{p_2^\rho}{p_2}\right)^{\frac{1}{\rho-1}} x_2 &= w \\ \Rightarrow x_2 &= \frac{w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \\ \Rightarrow x_1 &= \frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}. \end{aligned}$$

To find the indirect utility function, we must substitute the demand function back into the original utility function $u(\cdot)$. So,

$$\begin{aligned} v(p, w) &= \left(\left(\frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \right)^\rho + \left(\frac{w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \right)^\rho \right)^{\frac{1}{\rho}} \\ &= \left(w^\rho \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \right) \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \right)^{-\rho} \right)^{\frac{1}{\rho}} \\ &= w \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \right)^{\frac{1-\rho}{\rho}} = \frac{w}{\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}}. \end{aligned}$$

- (d) For CES demand function,

$$\frac{x_1(p, w)}{x_2(p, w)} = \frac{\frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}}{\frac{w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}.$$

So,

$$\begin{aligned} \xi_{12}(p, w) &= -\frac{\partial(x_1/x_2)}{\partial(p_1/p_2)} \left(\frac{p_1/p_2}{x_1(p, w)/x_2(p, 2)} \right) \\ &= -\frac{1}{\rho-1} \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}-1} \left(\frac{\frac{p_1}{p_2}}{\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}} \right) = \frac{1}{1-\rho}. \end{aligned}$$

For Leontief demand function,

$$\frac{x_1(p, w)}{x_2(p, w)} = 1.$$

Since this is constant, $\xi_{12}(p, w) = 0$.

For Cobb-douglas demand function,

$$\begin{aligned} \frac{x_1(p, w)}{x_2(p, w)} &= \frac{\frac{\alpha w}{p_1}}{\frac{\beta w}{p_2}} = \frac{\alpha}{\beta} \left(\frac{p_1}{p_2} \right)^{-1} \\ \Rightarrow \xi_{12}(p, w) &= -\frac{\partial(x_1/x_2)}{\partial(p_1/p_2)} \left(\frac{p_1/p_2}{x_1(p, w)/x_2(p, w)} \right) \\ &= -\frac{-\alpha}{\beta} \left(\frac{p_1}{p_2} \right)^{-2} \left(\frac{\frac{p_1}{p_2}}{\frac{\alpha}{\beta} \left(\frac{p_1}{p_2} \right)^{-1}} \right) = 1. \end{aligned}$$

4. An individual's labor supply decision (how many hours a day to work) can be modeled in the following way. There are two goods, leisure time and a consumption good. The individual has a utility function over leisure (L) and consumption (Y), $u(L, Y)$. The time not spent on leisure is the time spent on work. The individual earns income from working and uses it to buy the consumption good. Thus, letting w be the hourly wage rate and p be the price of the consumption good, the individual's budget constraint is $pY \leq w(24 - L)$, or equivalently $wL + pY \leq 24w$.

- (a) Suppose the individual's utility function is $U(L, Y) = L^\alpha Y$. Suppose the individual works 8 hours when $w = 5$ and $p = 1$. What is the value of α ?

Solution: The utility maximization problem is given by

$$\max_{L, Y} L^\alpha Y \quad \text{s.t.} \quad wL + pY = 24w$$

Since $p = 1$, this simplifies to

$$\max_{L, Y} L^\alpha Y \quad \text{s.t.} \quad wL + Y = 24w$$

Recognizing that the utility function is a transformation of standard Cobb-Douglas utility function,

$$u(L, Y) = L^{\frac{\alpha}{\alpha+1}} Y^{\frac{1}{\alpha+1}},$$

we obtain the (Marshallian) demand as

$$\begin{aligned} L(w) &= \frac{\alpha}{\alpha+1} \left(\frac{24w}{w} \right) = \frac{24\alpha}{\alpha+1} \\ Y(w) &= \frac{1}{\alpha+1} \left(\frac{24w}{1} \right) = \frac{24w}{\alpha+1}. \end{aligned}$$

Since Betty is working 8 hours, we have

$$8 = 24 - L^* = 24 - \frac{24\alpha}{\alpha+1} \implies 24\alpha = 16(\alpha+1) \implies 8\alpha = 16 \implies \alpha = 2.$$

In the following, use the value of α found in part (a).

- (b) Suppose Betty is given an additional \$40 lump-sum payment as a bonus. Find Betty's optimal consumption of leisure and the composite good. Will Betty work more or less than before?

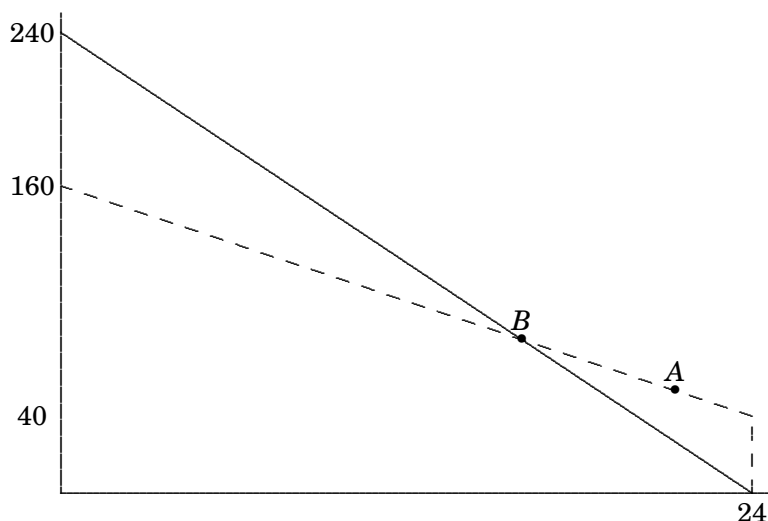
Solution: Receiving \$40 lump-sum bonus leads to the dashed-line budget set below.

$$\max_{L,Y} L^2 Y \quad \text{s.t.} \quad wL + Y = 40 + 24w$$

$$\Rightarrow \hat{L}(w) = \frac{2}{3} \left(\frac{40 + 24w}{w} \right) = \frac{2}{3} \left(\frac{40 + 24(5)}{5} \right) = \frac{2(160)}{3(5)} = 21.33$$

$$\hat{Y}(w) = \frac{1}{3} \left(\frac{40 + 24w}{1} \right) = \frac{160}{3} = 53.33,$$

which is point A on the graph below. Since $24 - 21.33 = 2.67 < 8$, Betty works fewer hours than before.



- (c) Will Betty be happier if her wage rate doubles to $w' = 10$ instead of getting the \$40 lump-sum bonus?

Solution: Doubling the wage rate yields the solid-line budget set above. Betty's optimal consumption is

$$L(w') = \frac{24\alpha}{\alpha + 1} = \frac{24(2)}{2 + 1} = 16$$

$$Y(w') = \frac{24w'}{\alpha + 1} = \frac{24(10)}{2 + 1} = 80,$$

which is point B on the graph above.

To see which option Betty prefers,

$$U(A) = (21.33)^2(53.33) = 24,263.49$$

$$U(B) = (16)^2(80) = 20,480$$

So Betty prefers to receive the \$40 bonus.

5. Let $u(x) = \min \{2x_1, x_2\}$.

*** As announced in class and on the course website, this question has been moved to Problem Set 2. ***

- (a) Find the Hicksian demand.
- (b) Find the expenditure function.
- (c) Verify that the expenditure function is concave in p .
- (d) Find the indirect utility function using the duality relationship.
- (e) Find the Marshallian demand.