Advanced Microeconomics I

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Problem Set 6: Suggested Solutions

1. Consider an economy with two goods, two consumers and one producer, where the consumers' utility functions and endowments are

$$u_1(x_{11}, x_{21}) = \ln x_{11} + \ln x_{21} \qquad \omega_1 = (10, 10)$$
$$u_2(x_{12}, x_{22}) = \ln x_{12} + 3\ln x_{22} \qquad \omega_2 = (10, 10).$$

The firm uses good 1 as input to produce good 2. Its production function is $f(z) = z^{\frac{1}{2}}$. The consumers' shares of the firm are given by $\theta_1 = \frac{1}{4}$ and $\theta_2 = \frac{3}{4}$. Find the Walrasian equilibrium.

Solution: As before, the firm's profit maximization problem is:

$$\max_z \; p_2 z^{\frac{1}{2}} - wz \implies \text{FOC is:} \quad \frac{p_2}{2} z^{-\frac{1}{2}} - w = 0.$$

Solving the FOC and normalizing $p_2 = 1$, we obtain

$$z(p_2, w) = \frac{p_2^2}{4w^2} = \frac{1}{4w^2}$$
$$y(p_2, w) = \left(\frac{p_2^2}{4w^2}\right)^{\frac{1}{2}} = \frac{p_2}{2w} = \frac{1}{2w}$$
$$\pi(p_2, w) = p_2 \frac{p_2}{2w} - w\frac{p_2^2}{4w^2} = \frac{p_2^2}{4w} = \frac{1}{4w}.$$

For consumer's demand functions, we first transform the non-standard Cobb-Douglas utility functions we have into standard Cobb-Douglas utility functions.

$$u_1(x_{11}, x_{21}) = \ln x_{11} + \ln x_{21} \sim x_{11}^{\frac{1}{2}} x_{21}^{\frac{1}{2}}$$
$$u_2(x_{12}, x_{22}) = \ln x_{12} + 3\ln x_{22} \sim x_{12}^{\frac{1}{4}} x_{22}^{\frac{3}{4}}.$$

At price $p = (1, w)$, the consumers' wealth are

$$W_1 = (w,1)\cdot\omega_1 + \theta_1\pi(w,1) = w(10) + 1(10) + \frac{1}{4}\left(\frac{1}{4w}\right) = 10 + 10w + \frac{1}{16w}$$
$$W_2 = (w,1)\cdot\omega_2 + \theta_1\pi(w,1) = w(10) + 1(10) + \frac{3}{4}\left(\frac{1}{4w}\right) = 10 + 10w + \frac{3}{16w}.$$

Therefore the consumers' demand functions are

$$\begin{aligned}
x_{11}(w,1) &= \frac{W_1}{2w} \\
x_{21}(w,1) &= \frac{W_1}{2} = \frac{10+10w+\frac{1}{16w}}{2} = 5+5w+\frac{1}{32w} \\
x_{12}(w,1) &= \frac{W_2}{4w} \\
x_{22}(w,1) &= \frac{3W_2}{4} = \frac{3(10+10w+\frac{3}{16w})}{4} = \frac{15}{2}+\frac{15w}{2}+\frac{9}{64w}.
\end{aligned}$$

To find the equilibrium price, we solve for market clearing condition for good 2:

$$x_{21}(w,1)+x_{22}(w,1) = 5+5w+\frac{1}{32w}+\frac{15}{2}+\frac{15w}{2}+\frac{9}{64w} = 20+\frac{1}{2w} = \bar{\omega}_2+y(w,1),$$

which yields

$$\begin{aligned}
\frac{25}{2}+\frac{25w}{2}+\frac{11}{64w} &= 20+\frac{1}{2w} \\
800w+800w^2+11 &= 1280w+32 \\
800w^2-480w-21 &= 0 \\
\implies w^* &= \frac{12+\sqrt{186}}{40} \approx 0.64.
\end{aligned}$$

The equilibrium allocation $\left(x_{11}^*, x_{21}^*, x_{12}^* x_{22}^*, z^*, y^*\right)$ can be found by substituting $w^* = 0.64$ into the demand and supply functions found above.

2. Consider a 2×2 production model where the two firms' production functions are

$$f_1(z_{11},z_{21}) = z_{11}^{\frac{2}{3}} z_{21}^{\frac{1}{3}} \quad \text{and} \quad f_2(z_{12},z_{22}) = z_{12}^{\frac{1}{2}} z_{22}^{\frac{1}{2}}.$$

The aggregate factor endowments are $\bar{z}_1 = 10$ and $\bar{z}_2 = 10$, and the prices of the output goods are $p_1 = 10$ and $p_1 = 10$.

(a) Verify that the production of good 1 is relatively more intense than the production of good 2.

Solution: Recall that for a generic CRS Cobb-Douglas production function,

$$f_j(z_{1j},z_{2j}) = z_{1j}^{a_j} z_{2j}^{b_j},$$

we have

$$z_{1j}(w_1^*,w_2^*,1) = \left(\frac{a_j w_2}{b_j w_1}\right)^{b_j} q \quad \text{and} \quad z_{2j}(w_1^*,w_2^*,1) = \left(\frac{b_j w_1}{a_j w_2}\right)^{a_j} q.$$

$$\implies \frac{z_{1j}(w_1,w_2,1)}{z_{2j}(w_1,w_2,1)} = \frac{\left(\frac{a_j w_2}{b_j w_1}\right)^{b_j} q}{\left(\frac{b_j w_1}{a_j w_2}\right)^{a_j} q} = \left(\frac{a_j w_2}{b_j w_1}\right)^{b_j}\left(\frac{a_j w_2}{b_j w_1}\right)^{a_j} = \frac{a_j w_2}{b_j w_1},$$

since $a_j + b_j = 1$ in our case. Therefore,

$$\frac{z_{11}(w_1,w_2,1)}{z_{21}(w_1,w_2,1)} = \frac{\frac{2}{3}w_2}{\frac{1}{3}w_1} = \frac{2w_2}{w_1} > \frac{w_2}{w_1} = \frac{\frac{1}{2}w_2}{\frac{1}{2}w_1} = \frac{z_{12}(w_1,w_2,1)}{z_{22}(w_1,w_2,1)}.$$

(b) Find the equilibrium factor prices.

Solution: To find the candidates for factor market equilibrium prices, we solve the two zero-profit conditions. The unit cost function of a firm with standard Cobb-Douglas production function is

$$c_j(w_1,w_2,1) = \left[\left(\frac{a_j}{b_j}\right)^{b_j} + \left(\frac{b_j}{a_j}\right)^{a_j}\right] w_1^{a_j} w_2^{b_j}.$$

Thus, the zero profit condition for firm 1 yields

$$10 = \left[\left(\frac{\frac{2}{3}}{\frac{1}{3}}\right)^{\frac{1}{3}} + \left(\frac{\frac{1}{3}}{\frac{2}{3}}\right)^{\frac{2}{3}}\right] w_1^{\frac{2}{3}} w_2^{\frac{1}{3}} = \left[2^{\frac{1}{3}} + \left(\frac{1}{2}\right)^{\frac{2}{3}}\right] w_1^{\frac{2}{3}} w_2^{\frac{1}{3}} = 1.890\, w_1^{\frac{2}{3}} w_2^{\frac{1}{3}}$$

$$\implies w_1^2 w_2 = \left(\frac{10}{1.890}\right)^3 = 148.12 \implies w_2 = \frac{148.12}{w_1^2}.$$

The zero profit condition for firm 2 yields

$$10 = \left[\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right)^{\frac{1}{2}} + \left(\frac{\frac{1}{2}}{\frac{1}{2}}\right)^{\frac{1}{2}}\right] w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} = 2 w_1^{\frac{1}{2}} w_2^{\frac{1}{2}}$$

$$\implies w_1 w_2 = \left(\frac{10}{2}\right)^2 = 25 \implies w_1\left(\frac{148.12}{w_1^2}\right) = 25$$

$$w_1^* = \left(\frac{148.12}{25}\right) = 5.92$$

$$w_2^* = \frac{148.12}{5.92^2} = 4.23.$$

To be sure that this the equilibrium, we need to check that the ratio of the aggregate factor endowments. $\frac{\bar{z}_1}{\bar{z}_2} = \frac{10}{10} = 1$ fall between the two firms' factor intensity ratios at these prices:

$$\frac{z_{11}(w_1^*,w_2^*,1)}{z_{21}(w_1^*,w_2^*,1)} = \frac{2w_2^*}{w_1^*} = \frac{2(4.23)}{5.92} = 1.43 > 1 > 0.71 = \frac{4.23}{5.92} = \frac{w_2^*}{w_1^*} = \frac{z_{12}(w_1^*,w_2^*,1)}{z_{22}(w_1^*,w_2^*,1)},$$

as required.

(c) Find the equilibrium factor allocation. What are the output levels in the equilibrium?

Solution: Using the conditional factor demand from part (a), we have,

$$z_{11}(w_1^*, w_2^*, 1) = \left(\frac{\frac{2}{3}(4.23)}{\frac{1}{3}(5.92)}\right)^{\frac{1}{3}} = \left(\frac{2(4.23)}{5.92}\right)^{\frac{1}{3}} = 1.126$$

$$z_{21}(w_1^*, w_2^*, 1) = \left(\frac{\frac{1}{3}(5.92)}{\frac{2}{3}(4.23)}\right)^{\frac{2}{3}} = \left(\frac{5.92}{2(4.23)}\right)^{\frac{2}{3}} = 0.788$$

$$z_{12}(w_1^*, w_2^*, 1) = \left(\frac{\frac{1}{2}(4.23)}{\frac{1}{2}(5.92)}\right)^{\frac{1}{2}} = \left(\frac{4.23}{5.92}\right)^{\frac{1}{2}} = 0.845$$

$$z_{22}(w_1^*, w_2, 1) = \left(\frac{\frac{1}{2}(5.92)}{\frac{1}{2}(4.23)}\right)^{\frac{1}{2}} = \left(\frac{5.92}{4.23}\right)^{\frac{1}{2}} = 1.183.$$

Next, we find the scaling required to clear the factor markets. That is we find, $\alpha > 0$ and $\beta > 0$ so that $\alpha z(w_1^*, w_2^*, 1) + \beta z(w_1^*, w_2^*, 1) = \bar{z}$. We write this as a column vector equation to make the scaling clearer.

$$\alpha \begin{bmatrix} 1.126 \\ 0.788 \end{bmatrix} + \beta \begin{bmatrix} 0.845 \\ 1.183 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

The first equation yields

$$\alpha = \frac{10 - 0.845\beta}{1.126} = 8.881 - 0.750\beta.$$

Substitute this into the second equation to obtain

$$\begin{aligned} &(8.881 - 0.750\beta)(0.788) + 1.183\beta = 10 \\ &\implies \beta = \frac{10 - 8.881(0.788)}{1.183 - 0.750(0.788)} = \frac{3.002}{0.592} = 5.071 \\ &\alpha = 8.881 - 0.750(5.071) = 5.078. \end{aligned}$$

Therefore, the equilibrum output levels are $q_1^* = \alpha = 5.071$ and $q_2^* = \beta = 5.078$. And the equilibrium factor allocations are:

$$\begin{aligned} z_{11}(w_1^*, w_2^*, q_1^*) &= 5.071(1.126) = 5.710 \\ z_{21}(w_1^*, w_2^*, q_1^*) &= 5.071(0.788) = 3.996 \\ z_{12}(w_1^*, w_2^*, q_1^*) &= 5.078(0.845) = 4.291 \\ z_{22}(w_1^*, w_2^*, q_1^*) &= 5.078(1.183) = 6.007. \end{aligned}$$

To verify that the markets indeed clear, note that $z_{11}^* + z_{12}^* = 5.710 + 4.291 = 10.001 \approx \bar{z}_1$ and $z_{21}^* + z_{22}^* = 3.996 + 6.007 = 10.003 \approx 10 = \bar{z}_2$. The discrepancies are rounding errors.