Advanced Microeconomics I

Fall 2024 - M. Pak

Problem Set 5: Suggested Solutions

1. Consider an Edgeworth box economy where preferences are given by

$$u_1(x_{11}, x_{21}) = x_{11}x_{21} \quad \text{and} \quad u_2(x_{12}, x_{22}) = x_{12} + \ln x_{22},$$

and the initial endowments are

$$\omega_1 = (0, 1) \quad \text{and} \quad \omega_2 = (1, 1).$$

Find the Pareto set, using x_{11} as the parameter. (Be sure to pay attention to the boundary of the Edgeworth box, where slopes of the indifference curves being equal is not necessary for Pareto optimality.)

Solution: Interior Pareto optimal allocations occur where $MRS_1 = MRS_2$.

$$\frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}} \implies \frac{x_{21}}{x_{11}} = \frac{1}{\frac{1}{x_{22}}} \implies \frac{x_{21}}{x_{11}} = x_{22}.$$

Substituting $x_{22} = 2 - x_{21}$ and solving gives us

$$x_{21} = 2x_{11} - x_{11}x_{21} \implies x_{21} = \frac{2x_{11}}{1 + x_{11}}.$$

So, interior Pareto optimal allocations are

$$\left\{ \left(x_{11}, \frac{2x_{11}}{1 + x_{11}} \right), \left(1 - x_{11}, 2 - \frac{2x_{11}}{1 + x_{11}} \right) \ : \ 0 \leq x_{11} \leq 1 \right\}.$$

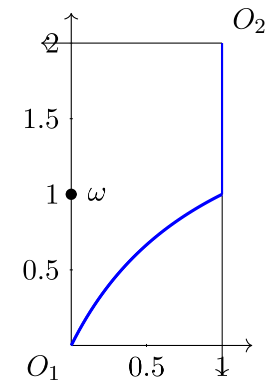
To find the boundary Pareto otimal allocations, note that when $x_{11} = 0$ or $x_{21} = 0$, consumer 1's utility is zero. So, any allocation on the left or the bottom edge of the Edgeworth box (other than the corner point, $((0, 0), (1, 2))$) cannot be Pareto optimal since consumer 2 can be made strictly better off without making consumer 1 worse off by moving to the allocation $((0, 0), (1, 2))$.

Next, the allocations in the top half of the right edge,

$$\{(1, x_{21}), (0, 2 - x_{21}) \ : \ 1 \leq x_{21} \leq 2\},$$

are Pareto optimal since at these allocations, consumer 1's indifference curve is steeper than the consumer 2's indifference curve. So, these allocations are Pareto optimal.

At allocations in the top edge, consumer 1's indifference curve is steeper than consumer 2's. At allocations in the bottom half of the right edge, consumer 1's indifference curves are flatter than consumer 2's. So, these allocations are not Pareto optimal since there are exact allocations that make one consumer strictly better off without making the other consumer worse off.



2. (MWG 15.B.2) Consider an Edgeworth box economy in which consumers have the Cobb-Douglas utility functions $u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha}$ and $u_2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta}$. Consumer i's endowments are $(\omega_{1i}, \omega_{2i}) \gg 0$, for $i = 1, 2$. Solve for the equilibrium price ratio and allocation. How do these change with differential change in ω_{11}?.

   Solution: Normalize by setting $p_2 = 1$. Then solving the utility maximization problem yields the following demand functions:

$$x_{11}(p) = \frac{\alpha(p_1\omega_{11} + \omega_{21})}{p_1} = \alpha\omega_{11} + \frac{\alpha\omega_{21}}{p_1}$$
$$x_{21}(p) = (1-\alpha)(p_1\omega_{11} + \omega_{21})$$
$$x_{12}(p) = \frac{\beta(p_1\omega_{12} + \omega_{22})}{p_1} = \beta\omega_{12} + \frac{\beta\omega_{22}}{p_1}$$
$$x_{22}(p) = (1-\beta)(p_1\omega_{12} + \omega_{22}).$$

   To find the equilibrium price, we use the market clearing condition for good 2:

$$x_{21}(p^*) + x_{22}(p^*) = \omega_{21} + \omega_{22} \iff (1-\alpha)(p_1^*\omega_{11} + \omega_{21}) + (1-\beta)(p_1^*\omega_{12} + \omega_{22}) = \omega_{21} + \omega_{22}$$
$$\iff p_1^*\big((1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\big) = \alpha\omega_{21} + \beta\omega_{22}$$
$$\iff p_1^* = \frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}.$$

   The equilibrium allocations are found by substituting p^* into the demand functions:

$$x_{11}^* = \alpha\omega_{11} + \frac{\alpha\omega_{21}}{p_1^*} = \alpha\omega_{11} + \frac{\alpha\omega_{21}\big((1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\big)}{\alpha\omega_{21} + \beta\omega_{22}}$$
$$x_{21}^* = (1-\alpha)(p_1\omega_{11} + \omega_{21}) = (1-\alpha)\omega_{11}\left(\frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}\right) + (1-\alpha)\omega_{21}$$
$$x_{12}^* = \beta\omega_{12} + \frac{\beta\omega_{22}}{p_1^*} = \beta\omega_{12} + \frac{\beta\omega_{22}\big((1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\big)}{\alpha\omega_{21} + \beta\omega_{22}}$$
$$x_{22}^* = (1-\beta)(p_1\omega_{12} + \omega_{22}) = (1-\beta)\omega_{12}\left(\frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}\right) + (1-\beta)\omega_{21}.$$

Thus, we have:

$$\frac{\partial p_1^*}{\partial \omega_{11}} = -\frac{(1-\alpha)(\alpha\omega_{21} + \beta\omega_{22})}{\big((1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\big)^2} < 0$$
$$\frac{\partial x_{11}^*}{\partial \omega_{11}} = \alpha + \frac{\alpha\omega_{21}(1-\alpha)}{\alpha\omega_{21} + \beta\omega_{22}} > 0$$
$$\frac{\partial x_{21}^*}{\partial \omega_{11}} > 0 \quad \text{by carefully differentiating}$$
$$\frac{\partial x_{12}^*}{\partial \omega_{11}} > 0 \quad \text{by carefully differentiating}$$
$$\frac{\partial x_{22}^*}{\partial \omega_{11}} = \frac{\partial}{\partial \omega_{11}}\big[\omega_{21} + \omega_{22} - x_{21}^*\big] = -\frac{\partial x_{21}^*}{\partial \omega_{11}} < 0.$$

3. Consider an Edgeworth box economy where preferences are given by

$$u_1(x_{11}, x_{21}) = x_{11} + \ln x_{21} \quad \text{and} \quad u_2(x_{12}, x_{22}) = x_{12}x_{22},$$

and the initial endowments are

$$\omega_1 = (1, 3) \quad \text{and} \quad \omega_2 = (3, 1).$$

(a) Using the normalization $p_2 = 1$, Find all the Walrasian equilibrium. You may assume that the solution is interior.

Solution: To find consumer 1's Marshallian demand, we solve

$$\max_{x_{11}, x_{21}} \; x_{11} + \ln x_{21} \quad \text{s.t.} \quad p_1x_{11} + p_2x_{21} = p_1 + 3p_2$$

Setting MRS = price ratio yields

$$x_{21} = \frac{p_1}{p_2}.$$

Substituting this into (3) yields:

$$p_1x_{11} + p_2\left(\frac{p_1}{p_2}\right) = p_1 + 3p_2 \quad \Rightarrow \quad x_{11} = \frac{3p_2}{p_1}.$$

Note that this also shows that there are no boundary solutions in this case. So, Marshallian demand for consumer 1 is:

$$x_1(p_1, p_2) = \left(\frac{3p_2}{p_1}, \frac{p_1}{p_2} \right).$$

To find consumer 2's Marshallian demand function, we note that her utility function is a Cobb-Douglas utility function, so the demand function is given by

$$x_2(p_1, p_2) = \left(\frac{3p_1 + p_2}{2p_1}, \frac{3p_1 + p_2}{2p_2} \right).$$

Now, we find the equilibrium prices by normalizing $p_2 = 1$ and looking for market clearing prices for the market for good 2:

$$x_{21}(p_1^*, 1) + x_{22}(p_1^*, 1) = \omega_{21} + \omega_{22} \implies \frac{p_1^*}{1} + \frac{3p_1^* + 1}{2} = 4$$
$$\implies p_1^* = \frac{7}{5}.$$

So, the Walrasian equilibrium price vector is $p^* = (\frac{7}{5}, 1)$, and the corresponding Walrasian equilibrium allocation is:

$$x_1^* = \left(\frac{15}{7}, \frac{7}{5} \right) \quad \text{and} \quad x_2^* = \left(\frac{13}{7}, \frac{13}{5} \right).$$

(b) Verify that the first welfare theorem holds.

Solution: At the Walrasian equilibrium allocation, we have

$$\text{MRS}_1\big|_{x_1^*} = \frac{x_{21}^*}{1} = \frac{7}{5} \quad \text{and} \quad \text{MRS}_2\big|_{x_1^*} = \frac{x_{22}^*}{x_{12}^*} = \frac{\frac{13}{5}}{\frac{13}{7}} = \frac{7}{5}.$$

Since $\text{MRS}_1\big|_{x_1^*} = \text{MRS}_2\big|_{x_2^*}$, (x_1^*, x_2^*) is Pareto optimal, as required.

(c) Can the allocation $\hat{x} = (\hat{x}_1, \hat{x}_2) = ((1,1),(3,3))$ be supported as a Walrasian equilibrium with transfers? If yes, find the supporting prices and transfers. If no, explain.

Solution: At allocation $\hat{x}$, we have

$$MRS_1 = \hat{x}_{21} = 1$$
$$MRS_2 = \frac{\hat{x}_{22}}{\hat{x}_{12}} = \frac{3}{3} = 1$$
$$\implies MRS_1 = MRS_2.$$

Thus, $\hat{x}$ is Pareto optimal, and the Second Welfare Theorem implies that we can support this allocation as an equilibrium with transfers. The supporting prices are given by the MRS of the consumers at this allocation. That is, $\frac{\hat{p}_1}{\hat{p}_2} = 1$. With our normalization, we get $\hat{p} = (1, 1)$. So, for transfers,

$$T_1 = (\hat{p}_1, \hat{p}_2) \cdot (\hat{x}_{11}, \hat{x}_{21}) - (\hat{p}_1, \hat{p}_2) \cdot (\omega_{11}, \omega_{21})$$
$$= (1,1) \cdot (1,1) - (1,1) \cdot (1,3) = -2$$
$$T_2 = -T_1 = 2.$$

4. Consider a pure exchange economy with 2 consumers and 2 goods, where the consumers' preferences and endowments are given by

$$\begin{aligned} u_1(x_{11}, x_{21}) &= \min\{x_{11}, x_{21}\} \qquad \omega_1 = (6,5) \\ u_2(x_{12}, x_{22}) &= \min\{x_{12}, x_{22}\} \qquad \omega_2 = (6,5). \end{aligned}$$

(a) Using a carefully labeled Edgeworth box diagram, graph the PO Set.

Solution: Pareto set is the thick area between the two dashed lines.

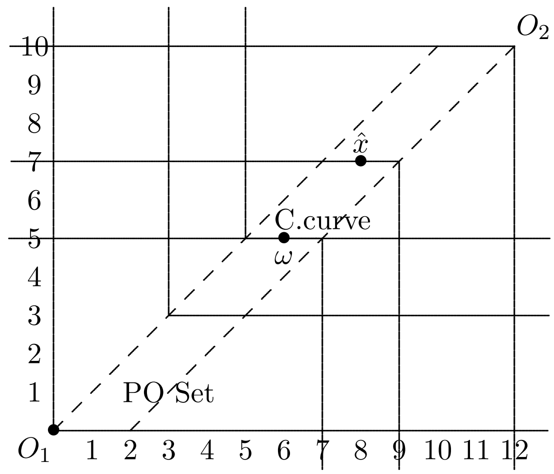


Figure 1: Pareto Set.

(b) Using a carefully labeled Edgeworth box diagram, graph the contract curve.

Solution: Contract curve is the portion of Pareto set where every consumer is doing as well as their endowment. This is the short line segment in the Pareto set that goes through the endowment point.

(c) Can allocation $(\hat{x}_1, \hat{x}_2) = \big((8,7),(4,3)\big)$ be supported as a Walrasian equilibrium with transfers? If so, find the supporting prices and the corresponding transfers. If not, explain.

Solution: Yes, it is possible to support $\hat{x}$ as equilibrium with transfers. Supporting price for $\hat{x}$ is $\hat{p} = (0,1)$. The required transfers are

$$\begin{aligned} T_1 &= \hat{p} \cdot \hat{x}_1 - \hat{p} \cdot \omega_1 = (0,1) \cdot (8,7) - (0,1) \cdot (6,5) = 7 - 5 = 2 \\ T_2 &= -2. \end{aligned}$$

5. Consider a Robinson Crusoe economy where the consumer's utility function is $u(x_1, x_2) = 2\ln x_1 + \ln x_2$ and the production function is $f(z) = z^{\frac{1}{2}}$. Suppose the consumer's endowment is $\omega = (5,0)$.

(a) Letting w and p_2 be the prices of good 1 and good 2, respectively, find the firm's unconditional input demand, supply, and profit functions.

Solution: The firm's profit maximization problem is:

$$\max_{z}\ p_2 z^{\frac{1}{2}} - wz$$

$$\text{FOC is:}\quad \frac{p_2}{2} z^{-\frac{1}{2}} - w = 0 \quad \Longrightarrow \quad z(p_2, w) = \frac{p_2^2}{4w^2}$$

$$y(p_2, w) = \left(\frac{p_2^2}{4w^2}\right)^{\frac{1}{2}} = \frac{p_2}{2w}$$

$$\pi(p_2, w) = p_2 \frac{p_2}{2w} - w \frac{p_2^2}{4w^2} = \frac{p_2^2}{4w}.$$

(b) As usual, assume that all the profits of the firm goes to the consumer in this economy and find find the consumer's demand function.

Solution: In this example, the consumer's demand can be easily found by transforming the utility function into a standard Cobb-Douglas form and applying the usual formula. However, as a practice, we'll find the demand by actually solving the optimization problem. The consumer's problem is:

$$\max_{x_1, x_2}\ 2\ln x_1 + \ln x_2 \qquad \text{s.t.} \qquad wx_1 + p_2 x_2 \leq w\omega_1 + p_2\omega_2 + \pi(p_2, w).$$

$$\Longrightarrow \qquad \mathcal{L} = 2\ln x_1 + \ln x_2 + \lambda\left(5w + \frac{p_2^2}{4w} - wx_1 - p_2 x_2\right).$$

Assuming interior solution:

$$(1) \qquad \frac{\partial \mathcal{L}}{\partial x_1} = \frac{2}{x_1} - \lambda w = 0$$

$$(2) \qquad \frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{x_2} - \lambda p_2 = 0$$

$$(3) \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 5w + \frac{p_2^2}{4w} - wx_1 - p_2 x_2 = 0.$$

From (1) and (2) we can get:

$$\frac{2x_2}{x_1} = \frac{w}{p_2} \Longrightarrow x_1 = \frac{2p}{w} x_2$$

Substituting this into (3) yields:

$$w\left(\frac{2p}{w} x_2\right) + p_2 x_2 = 5w + \frac{p_2^2}{4w}$$

$$\Longrightarrow \qquad x_2(w, p_2) = \frac{5w + \frac{p_2^2}{4w}}{3p} \qquad \text{and} \qquad x_1(w, p_2) = \frac{2\left(5w + \frac{p_2^2}{4w}\right)}{3w}.$$

(c) Suppose prices are $(w, p_2) = (1, 1)$. Using a single diagram, graph the consumer's utility-maximizing consumption bundle and the firm's profit maximizing production plan. Is this a Walrasian equilibrium price vector? Explain.

Solution: When $(w, p_2) = (1,1)$, $z(1,1) = \frac{1}{4}$ and $y(1,1) = \frac{1}{2}$, while $x_1(1,1) = \frac{7}{2}$ and $x_2(1,1) = \frac{7}{4}$. The graph of the economy is given in figure 2. As this figure shows, at prices $(w, p_2) = (1,1)$, markets do not clear. For example,

$$x_2(1,1) = \frac{7}{4} \neq 0 + \frac{1}{2} = \omega_2 + y(1,1).$$

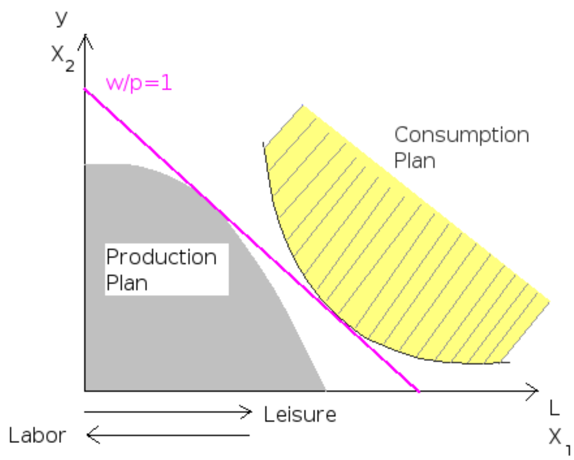


Figure 2: Production and Consumption Plan at $(w, p_2) = (1,1)$

(d) Find all the Walrasian equilibrium using the normalization $p_2 = 1$.

Solution: We solve for the market clearing condition for good 2:

$$x_2(w,1) = \omega_2 + y(w,1) \quad \Longrightarrow \quad \frac{5w + \frac{1}{4w}}{3} = \frac{1}{2w} \quad \Longrightarrow \quad w^* = \frac{1}{2}.$$

Thus, Walrasian equilibrium price is $(w^*, p_2^*) = (\frac{1}{2}, 1)$, and the corresponding Walrasian equilibrium consumption and production plans are $(x_1^*, x_2^*) = (4,1)$ and $(z^*, y^*) = (1,1)$.