Advanced Microeconomics I

Fall 2024 - M. Pak

Problem Set 4: Suggested Solutions

1. Show that $c(F, u_2) \leq c(F, u_1)$ for all distributions F if and only if there exists an increasing concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1(x))$. (Hint: use Jensen's inequality).

   Solution: Any concave function $g(\cdot)$ satisfies Jensen's inequality:

$$\int g(x)\, dF(x) \leq g\left(\int x\, dF(x)\right).$$

   For any distribution F, we have

$$\begin{aligned}
u_2\left(c(F, u_2)\right) &= \int u_2(x)\, dF \quad \text{by the definition of certainty equivalent} \\
&= \int \phi(u_1(x))\, dF \\
&\leq \phi\left(\int u_1(x)\, dF\right) \quad \text{by Jensen's inequality} \\
&= \phi\left(u_1(c(F, u_1))\right) \quad \text{by the definition of certainty equivalent} \\
&= u_2(c(F, u_1)).
\end{aligned}$$

   Since $u_2\left(c(F, u_2)\right) \leq u_2\left(c(F, u_1)\right)$ and $u_2(\cdot)$ is increasing, we have $c(F, u_2) \leq c(F, u_1)$ as desired.

2. A strictly risk-averse decision maker has initial wealth W and faces possible loss of $D < W$. The probability that the loss will occur is $\frac{1}{2}$. Suppose insurance is available at price $q < 1$ per unit, where q is not necessarily the fair price.

   (a) For what values of q will the decision maker want to buy strictly positive amount of insurance?

   Solution: The DM's expected utility is

$$\begin{aligned}
U(x) &= \frac{1}{2}u(W - qx) + \frac{1}{2}u(W - D - qx + x) \\
&= \frac{1}{2}u(W - qx) + \frac{1}{2}u(W - D + (1-q)x). \\
\Rightarrow \quad MU(x) &= \frac{1}{2}u'(W - qx)(-q) + \frac{1}{2}u'(W - D + (1-q)x)(1-q).
\end{aligned}$$

For $x^* > 0$, we need require $MU(0) > 0$:

$$\frac{1}{2}u'(W)(-q) + \frac{1}{2}u'(W-D)(1-q) > 0$$
$$\Rightarrow \quad q < \frac{u'(W-D)}{u'(W)+u'(W-D)}.$$

(b) Suppose the decision maker exhibits decreasing absolute risk aversion. Assuming that the optimal insurance purchase x^* is an interior solution (that is, $x^* \in (0, D)$) determine whether x^* is increasing or decreasing as the initial wealth, W, increases.

Solution: FOC for interior solution is given by

$$\frac{1}{2}u'(W-qx^*)(-q) + \frac{1}{2}u'(W-D+(1-q)x^*)(1-q) = 0.$$

Differentiating this w.r.t. W yields

$$u''(W-qx^*)(-q) + u''(W-qx^*)(-q)^2\frac{dx^*}{dW} + u''(W-D+(1-q)x^*)(1-q)$$
$$+u''(W-D+(1-q)x^*)(1-q)^2\frac{dx^*}{dW} = 0$$
$$\Longrightarrow \frac{dx^*}{dW} = \frac{-u''(W-qx^*)(-q) - u''(W-D+(1-q)x^*)(1-q)}{u''(W-qx^*)(-q)^2 + u''(W-D+(1-q)x^*)(1-q)^2}.$$

Note that $x^* < D$ and DARA means

$$W-D+(1-q)x^* = W-qx^*+(x^*-D) < W-qx^* \Longrightarrow R_a(W-D+(1-q)x^*) > R_a(W-qx^*).$$

Therefore, the numerator is:

$$\begin{aligned}
& -u''(W-qx^*)(-q) - u''(W-D+(1-q)x^*)(1-q) \\
&= -\frac{u''(W-qx^*)}{u'(W-qx^*)}u'(W-qx^*)(-q) - \frac{u''(W-D+(1-q)x^*)}{u'(W-D+(1-q)x^*)}u'(W-D+(1-q)x^*)(1-q) \\
&= R_a(W-qx^*)u'(W-qx^*)(-q) + R_a(W-D+(1-q)x^*)u'(W-D+(1-q)x^*)(1-q) \\
&> R_a(W-qx^*)u'(W-qx^*)(-q) + R_a(W-qx^*)u'(W-D+(1-q)x^*)(1-q) \\
&= R_a(W-qx^*)\underbrace{\Big(u'(W-qx^*)(-q) + u'(W-D+(1-q)x^*)(1-q)\Big)}_{=0 \text{ by FOC}} = 0.
\end{aligned}$$

Thus, we have $\frac{dx^*}{dW} = \frac{(+)}{(-)} < 0$.

3. Consider an investor whose utility function over money is

$$u(w) = 2w^{\frac{1}{2}}.$$

The investor can invest in a riskless asset that returns 1 (gross return per ¥1 invested) for sure, or a risky asset that returns 1.4 with probability $\frac{3}{4}$ and 0.8 with probability $\frac{1}{4}$.

(a) Suppose the investor's initial wealth is ¥1000. Letting x denote the amount invested in the risky asset, write the investor's expected utility as a function of x.

Solution: We have

$$g(x) = \begin{cases} 1000 - x + 1.4x = 1000 + 0.4x & \text{with probability } \frac{3}{4} \\ 1000 - x + 0.8x = 1000 - 0.2x & \text{with probability } \frac{1}{4} \end{cases}$$

So,

$$\begin{aligned} U(g(x)) &= \frac{3}{4}u(1000 + 0.4x) + \frac{1}{4}u(1000 - 0.2x) \\ &= \frac{3}{4}\left(2(1000 + 0.4x)^{\frac{1}{2}}\right) + \frac{1}{4}\left(2(1000 - 0.2x)^{\frac{1}{2}}\right) \end{aligned}$$

(b) Find the optimal amount to invest in the risky asset.

Solution: Investor solves:

$$\max_x \frac{3}{4}\left(2(1000 + 0.4x)^{\frac{1}{2}}\right) + \frac{1}{4}\left(2(1000 - 0.2x)^{\frac{1}{2}}\right)$$

FOC is given by

$$\left(\frac{3}{4}\right)\left(\frac{4}{10}\right)(1000 + 0.4x)^{-\frac{1}{2}} - \left(\frac{1}{4}\right)\left(\frac{2}{10}\right)(1000 - 0.2x)^{-\frac{1}{2}} = 0$$

$$\begin{aligned} \Rightarrow \left(\frac{3}{10}\right)(1000 + 0.4x)^{-\frac{1}{2}} &= \left(\frac{1}{20}\right)(1000 - 0.2x)^{-\frac{1}{2}} \\ \Rightarrow \left(\frac{10}{3}\right)^2 (1000 + 0.4x) &= (20)^2 (1000 - 0.2x) \\ \Rightarrow x &= \frac{(400)(1000) - \left(\frac{100}{9}\right)(1000)}{\left(\frac{100}{9}\right)(0.4) + (400)(0.2)} \approx 4605.263. \end{aligned}$$

Note that $x^* = 4605.263$ is greater than the initial wealth. So, if so-called "short sale" is possible, the investor will borrow additional \$3,605.263 to invest in the risky asset. If borrowing is not possible, then the investor will put the maximum amount in the risky asset. That is, $x^* = 1000$.

4. An investor with initial wealth w_0 is trying to allocate her wealth between a safe asset with constant return $R > 0$ and a risky asset with random return $\boldsymbol{z}$, where $\boldsymbol{z}$ has distribution function F and $\mathrm{E}[\boldsymbol{z}] > R$. Letting x be the *proportion* of wealth invested in the risky asset ($0 \le x \le 1$), her wealth will be:

$$w = \big((1 - x)R + xz\big)w_0 = \big(R + x(z - R)\big)w_0,$$

where z is the realized return. Suppose the investor's utility over wealth, $u(\cdot)$, exhibits constant relative risk aversion. Show that the optimal proportion of wealth invested in the risky asset is independent of her initial wealth. That is, show that $\frac{dx^*}{dw_0} = 0$.

Solution: The investor's expected utility maximization is

$$\max_x \mathrm{E}\big[u\big(w(x)\big)\big] \iff \max_x \int u\big(\big(R + x(z-R)\big)w_0\big)\, dF(z).$$

The first order condition is

$$\begin{aligned} & \int u'\big(\big(R + x^*(z-R)\big)w\big)(z-R)w\, dF(z) \;=\; 0 \\ \iff \quad & \int u'\big(\big(R + x^*(z-R)\big)w\big)(z-R)\, dF(z) \;=\; 0 \quad (*) \end{aligned}$$

Differentiating the FOC $(*)$ with respect to w yields (note that we use chain rule on x^*)

$$\begin{aligned} & \int u''\big(\big(R + x^*(z-R)\big)w\big)(z-R)^2 w \frac{dx^*}{dw}\, dF(z) \\ & \qquad + \int u''\big(\big(R + x^*(z-R)\big)w\big)(z-R)(R + x^*(z-R))\, dF(z) \;=\; 0 \end{aligned}$$

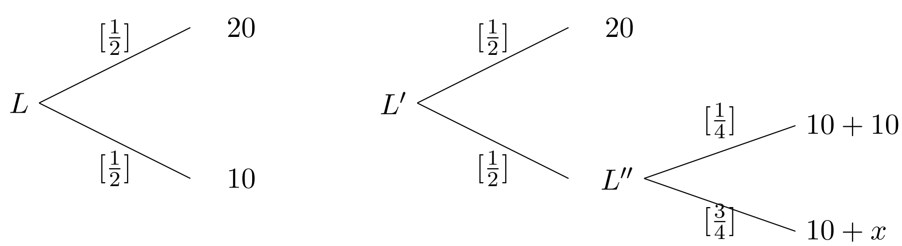
$$\Rightarrow \quad \frac{dx^*}{dw} \;=\; \frac{-\int u''\big(\big(R + x^*(z-R)\big)w\big)(z-R)(R + x^*(z-R))\, dF(z)}{\int u''\big(\big(R + x^*(z-R)\big)w\big)(z-R)^2 w\, dF(z)}$$

Looking at the numerator, we see that

$$\begin{aligned} & \int u''\big(\big(R + x^*(z-R)\big)w\big)(z-R)(R + x^*(z-R))\, dF(z) \\ = \; & \int \frac{wu''\big(\big(R + x^*(z-R)\big)w\big)(R + x^*(z-R))}{u'\big(\big(R + x^*(z-R)\big)w\big)} \left(\frac{u'\big(\big(R + x^*(z-R)\big)w\big)(z-R)}{w} \right) dF(z) \\ = \; & \frac{\text{constant}}{w} \int u'\big(\big(R + x^*(z-R)\big)w\big)(z-R)\, dF(z) \quad \text{by CRRA} \\ = \; & 0 \quad \text{by } (*). \end{aligned}$$

So, $\frac{dx^*}{dw} = 0$, as required.

5. Let L be the lottery $(\frac{1}{2} \circ 20,\ \frac{1}{2} \circ 10)$, and let L' be the lottery obtained from L in the following way. If the outcome 20 is realized under L, then nothing further happens. However, if the outcome 10 is realized, then another lottery $L'' = (\frac{1}{4} \circ 10,\ \frac{3}{4} \circ x)$ is drawn and the final outcome is the outcome from L plus the outcome from L'', as illustrated below.



(a) Find the values of x, if any, for which L' first order stochastically dominate L.

Solution: For L' to first order stochastically dominate L, we need $F_{L'} \leq F_L$. Since L' only takes values $10 + x$ or 20, it means that $x > 0$.

(b) Find the values of x, if any, for which L first order stochastically dominate L'.

Solution: For L to first order stochastically dominate L', we need $F_L(10) \leq F_{L'}(10)$. Since L' only takes values $10 + x$ or 20, it means that $x \leq 0$. However, then

$$F_L(15) = \frac{1}{2} \not\leq \frac{3}{8} = P(L' = 15).$$

So L cannot FOSD L' for any value of x.

(c) Find the values of x, if any, for which L second order stochastically dominate L'.

Solution: We need

$$E[L''] = \frac{10}{4} + \frac{3x}{4} \leq 0 \implies x \leq \frac{-10}{3}.$$