Advanced Microeconomics I

Fall 2024 - M. Pak

Problem Set 1: Suggested Solutions

1. Let $u(x) = ax_1 + bx_2^{\frac{1}{2}}$, where $a, b \in \mathbb{R}$, be a utility function representing a preference relation $\succsim$ on the consumption set $\mathbb{R}^2_+$.

   (a) Find the conditions on a and b under which $u(\cdot)$ is locally non-satiated, weakly monotone, and strongly monotone, respectively. Which of the three properties is the most restrictive? The least restrictive?

   Solution: If $a \leq 0$ and $b \leq 0$, then the utility is weakly decreasing in both x_1 and x_2, so there is no bundle that is better than $(0,0)$ in the consumption set. Thus, $\succsim$ is not locally non-satiated. But if $a > 0$, then for any $x = (x_1, x_2)$ and any neighborhood of radius $\varepsilon > 0$, we can find small enough γ such that $y = (x_1 + \gamma, x_2) \succ (x_1, x_2)$ and y is in that neighborhood. Similarly, if $b > 0$, then for any $x = (x_1, x_2)$ and any neighborhood of radius $\varepsilon > 0$, we can find small enough γ such that $y = (x_1, x_2 + \gamma) \succ (x_1, x_2)$ and y is in that neighborhood. Thus, $\succsim$ is locally non-satiated on $\mathbb{R}^2_+$ if and only if $a > 0$ or $b > 0$ (note that "or" means at least one of the two inequalities has to hold, so for example $a > 0$ and $b < 0$ is allowed). For (weak) monotonicity, we need $a \geq 0$ and $b \geq 0$ and at least one of the inequality to hold strictly. For strong monotonicity, we need $a > 0$ and $b > 0$. Thus, strong monotonicity is the most restrictive and local non-satiation is the least restrictive.

For the remainder of the questions, assume that $a = 1$, $b = 2$, and $p \gg 0$.

   (b) Find the Marshallian demand. (Please pay attention to the possibility of boundary solutions).

   Solution: Let

$$\mathscr{L}(x_1, x_2, \lambda) = x_1 + 2x_2^{\frac{1}{2}} + \lambda\big(w - p_1x_1 - p_2x_2\big).$$

   The first order conditions are:

$$(1) \quad \frac{\partial \mathscr{L}}{\partial x_1} = 1 - \lambda p_1 \leq 0 \quad (= 0 \text{ if } x_1 > 0)$$

$$(2) \quad \frac{\partial \mathscr{L}}{\partial x_2} = x_2^{-\frac{1}{2}} - \lambda p_2 \leq 0 \quad (= 0 \text{ if } x_2 > 0)$$

$$(3) \quad \frac{\partial \mathscr{L}}{\partial \lambda} = w - p_1x_1 - p_2x_2 = 0.$$

Assuming interior solution and dividing (1) by (2) yields

$$x_2^{\frac{1}{2}} = \frac{p_1}{p_2} \implies x_2^* = \left(\frac{p_1}{p_2}\right)^2.$$

Substitute this into the budget equation to obtain:

$$x_1^* = \frac{w - p_2\left(\frac{p_1}{p_2}\right)^2}{p_1} = \frac{w}{p_1} - \frac{p_1}{p_2}.$$

From the expression on x_1^* above, we see that an interior solution is obtained only if $\frac{p_1^2}{p_2} < w$. Thus, when $\frac{p_1^2}{p_2} \geq w$, the non-negativity constraint will force $x_1^* = 0$, and then the budget equation yields $x_2^* = \frac{w}{p_2}$. To summarize, the Marshallian demand (written as a row vector) is given by:

$$x(p,w) = \big(x_1(p,w),\ x_2(p,w)\big) = \begin{cases} \left(\frac{w}{p_1} - \frac{p_1}{p_2}\ ,\ \left(\frac{p_1}{p_2}\right)^2\right) & \text{if } \frac{p_1^2}{p_2} < w \\ \left(0\ ,\ \frac{w}{p_2}\right) & \text{if } \frac{p_1^2}{p_2} \geq w. \end{cases}$$

(c) Find the indirect utility function.

Solution: Since $v(p,w) = u(x(p,w)) = x_1(p,w) + 2x_2(p,w)^{\frac{1}{2}}$, we have

$$v(p,w) = \begin{cases} \frac{w}{p_1} - \frac{p_1}{p_2} + 2\left(\frac{p_1}{p_2}\right) & \text{if } \frac{p_1^2}{p_2} < w \\ 2\left(\frac{w}{p_2}\right)^{\frac{1}{2}} & \text{if } \frac{p_1^2}{p_2} \geq w. \end{cases}$$

2. Mas-Colell, Whinston, and Green (MWG) 3.C.6. For part (b), you may restrict attention to $x \gg 0$. In addition, it may be easier to work with the logarithmic transformation of the utility in part (b), $u(x) = \frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho}$, assuming that $\alpha_1 + \alpha_2 = 1$ (but explain why you are justified in doing this).

Solution: Let $u(x)$ be a CES utility function

$$u(x) \ = \ \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{\frac{1}{\rho}}$$

(a) As $\rho \to 1$, the utility function converges to $\alpha_1 x_1 + \alpha_2 x_2$, which is linear.

(b) Recall that a strictly increasing transformation of a utility function does not change the underlying preference. So we can assume $\alpha_1 + \alpha_2 = 1$ without loss of generality and can also use the natual log of the CES utility function:

$$u(x) = \frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho}$$

Since both the numerator and the denominator converges to zero as $\rho \to 0$, we can use L'Hopital's rule:

$$\begin{aligned}
\lim_{\rho \to 0} u(x) &= \lim_{\rho \to 0} \frac{\frac{\partial \ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\partial \rho}}{\frac{\partial \rho}{\partial \rho}} \\
&= \lim_{\rho \to 0} \frac{\alpha_1 x_1^\rho \ln x_1 + \alpha_2 x_2^\rho \ln x_2}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \\
&= \alpha_1 \ln x_1 + \alpha_2 \ln x_2.
\end{aligned}$$

This is a natural log transformation of a standard Cobb-Douglas utility function

$$u(x) = x_1^{\alpha_1} x_2^{\alpha_2}.$$

(c) We need to show that

$$\lim_{\rho \to -\infty} u(x) = \lim_{\rho \to -\infty} \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{\frac{1}{\rho}} = \min\{x_1, x_2\}.$$

Without loss of generality, we can let $x_1 = \min\{x_1, x_2\}$. Consider any $\rho < 0$. Since $x_1 \leq x_2$, we have

$$\begin{aligned}
\alpha_1 x_1^\rho &\leq \alpha_1 x_1^\rho + \alpha_2 x_2^\rho \leq \alpha_1 x_1^\rho + \alpha_2 x_1^\rho = (\alpha_1 + \alpha_2) x_1^\rho \\
\alpha_1^{\frac{1}{\rho}} x_1 &\geq (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} \geq (\alpha_1 + \alpha_2)^{\frac{1}{\rho}} x_1
\end{aligned}$$

Letting $\rho \to -\infty$, we obtain,

$$x_1 \geq \lim_{\rho \to -\infty} \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{\frac{1}{\rho}} \geq x_1,$$

as required.

3. MWG 3.D.5, parts (a) and (d).

Solution: Let $u(x) = \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}}$.

(a) To find the demand function, we may use the transformation $\tilde{u}(x) = (u(x))^\rho = x_1^\rho + x_2^\rho$.

The first order condition for an interior solution is:

$$
\begin{aligned}
\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} &= \frac{\rho x_1^{\rho-1}}{\rho x_2^{\rho-1}} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_1 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} x_2 \\
\Rightarrow \quad p_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} x_2 + p_2 x_2 &= w \\
\Rightarrow \quad \left(\frac{p_1^\rho}{p_2}\right)^{\frac{1}{\rho-1}} x_2 + \left(\frac{p_2^\rho}{p_2}\right)^{\frac{1}{\rho-1}} x_2 &= w \\
\Rightarrow \quad x_2 &= \frac{w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \\
\Rightarrow \quad x_1 &= \frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}.
\end{aligned}
$$

To find the indirect utility function, we must substitute the demand function back into the original utility function $u(\cdot)$. So,

$$
\begin{aligned}
v(p,w) &= \left(\left(\frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right)^\rho + \left(\frac{w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right)^\rho\right)^{\frac{1}{\rho}} \\
&= \left(w^\rho \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\rho}\right)^{\frac{1}{\rho}} \\
&= w\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} = \frac{w}{\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}}.
\end{aligned}
$$

(d) For CES demand function,

$$
\frac{x_1(p,w)}{x_2(p,w)} = \frac{\frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}}{\frac{w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}.
$$

So,

$$
\begin{aligned}
\xi_{12}(p,w) &= -\frac{\partial(x_1/x_2)}{\partial(p_1/p_2)}\left(\frac{p_1/p_2}{x_1(p,w)/x_2(p,2)}\right) \\
&= -\frac{1}{\rho-1}\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}-1}\left(\frac{\frac{p_1}{p_2}}{\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}}\right) = \frac{1}{1-\rho}.
\end{aligned}
$$

For Leontief demand function,

$$\frac{x_1(p,w)}{x_2(p,w)} = 1.$$

Since this is constant, $\xi_{12}(p,w) = 0$.
For Cobb-douglas demand function,

$$\begin{aligned}
\frac{x_1(p,w)}{x_2(p,w)} &= \frac{\frac{\alpha w}{p_1}}{\frac{\beta w}{p_2}} = \frac{\alpha}{\beta}\left(\frac{p_1}{p_2}\right)^{-1} \\
\Rightarrow \quad \xi_{12}(p,w) &= -\frac{\partial(x_1/x_2)}{\partial(p_1/p_2)}\left(\frac{p_1/p_2}{x_1(p,w)/x_2(p,2)}\right) \\
&= -\frac{-\alpha}{\beta}\left(\frac{p_1}{p_2}\right)^{-2}\left(\frac{\frac{p_1}{p_2}}{\frac{\alpha}{\beta}\left(\frac{p_1}{p_2}\right)^{-1}}\right) = 1.
\end{aligned}$$

4. Let $u(x) = \min\{2x_1, x_2\}$.

(a) Find the Hicksian demand.

Solution: Expenditure minimization problem is

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \quad \text{s.t.} \quad \min\{2x_1, x_2\} = u.$$

The solution occurs where

$$\begin{aligned}
&(1) \qquad 2x_1 = x_2, \quad \text{and} \\
&(2) \qquad \min\{2x_1, x_2\} = u.
\end{aligned}$$

Thus,

$$\begin{aligned}
x_1^h(p,u) &= \frac{u}{2}, \\
x_2^h(p,u) &= u.
\end{aligned}$$

(b) Find the expenditure function.

Solution:

$$e(p,u) = p \cdot x^h(p,u) = \frac{p_1u}{2} + p_2u.$$

(c) Verify that the expenditure function is concave in p.

Solution: Consider any two price vectors p and p' and a scalar $\alpha \in [0,1]$. We need to show that

$$e(\alpha p + (1-\alpha)p', u) \geq \alpha e(p,u) + (1-\alpha)e(p',u).$$

Since

$$\begin{aligned}
\alpha p + (1-\alpha)p' &= \alpha(p_1, p_2) + (1-\alpha)(p_1', p_2') \\
&= \left(\alpha p_1 + (1-\alpha)p_1'\,,\ \alpha p_2 + (1-\alpha)p_2'\right),
\end{aligned}$$

we have

$$\begin{aligned}
e(\alpha p+(1-\alpha)p',u) &= \frac{\left(\alpha p_1+(1-\alpha)p_1'\right)u}{2}+\left(\alpha p_2+(1-\alpha)p_2'\right)u \\
&= \alpha\left(\left(\frac{p_1 u}{2}\right)+p_2 u\right)+(1-\alpha)\left(\left(\frac{p_1' u}{2}\right)+p_2' u\right) \\
&= \alpha e(p,u)+(1-\alpha)e(p',u).
\end{aligned}$$

Notice that we get equality here since the two goods are perfect complements for the consumer so that the substitution effect is zero.

(d) Find the indirect utility function using the duality relationship.

solution From part (b), we have

$$e\big(p,v(p,w)\big)=w \iff \frac{p_1 v(p,w)}{2}+p_2 v(p,w)=w \iff \left(\frac{p_1}{2}+\frac{2p_2}{2}\right)v(p,w)=w$$

$$\implies v(p,w)=\frac{2w}{p_1+2p_2}.$$

(e) Find the Marshallian demand.

Solution We can find the Marshallian demand by solving the UMP. However, since we have the indirect utility function, it seems easier to find the Marshallian demand by using Roy's identity:

$$\begin{aligned}
x_1(p,w) &= -\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\frac{-2w}{(p_1+2p_2)^2}}{\frac{2}{p_1+2p_2}} = \frac{w}{p_1+2p_2} \\
x_2(p,w) &= -\frac{\frac{\partial v(p,w)}{\partial p_2}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\frac{-2w(2)}{(p_1+2p_2)^2}}{\frac{2}{p_1+2p_2}} = \frac{2w}{p_1+2p_2}.
\end{aligned}$$

5. A simple two-period consumption model can be constructed in the following way. An individual lives for two periods in an economy that has only one good. Letting x_t be the amount of the good the individual consumes in period t, she wants to maximize her lifetime utility:

$$u_i(x_1,x_2)=x_1^{\alpha}x_2^{\beta}, \quad \text{where } \alpha>0,\ \beta>0, \text{ and } \alpha+\beta=1.$$

The individual receives wealth w_t in period t, and she can borrow or save as much as she wants in period 1 at interest rate $r>0$. That is, if she saves s RMB in period 1, she will receive $(1+r)s$ RMB in period 2 (if $s<0$, then s is considered to be a borrowing). Letting p_t be the price of the good in period t, the individual's budget constraint in each period is:

$$\begin{aligned}
\text{t = 1:}\quad & p_1x_1=w_1-s \\
\text{t = 2:}\quad & p_2x_2=w_2+(1+r)s.
\end{aligned}$$

(a) Show that the budget constraint can be written as a single equation: $(1+r)p_1x_1+p_2x_2=(1+r)w_1+w_2$. Give an interpretation of the left and the right side of this equation.

Solution:

$$s = w_1 - p_1 x_1 \implies p_2 x_2 = w_2 + (1+r)(w_1 - p_1 x_1) \iff (1+r)p_1 x_1 + p_2 x_2 = (1+r)w_1 + w_2.$$

The left-hand side is the value of the consumption in future value and the right-hand side is the total lifetime income in future value.

For the remainder of the problem, assume that $p_1 = p_2 = 1$.

(b) Find the conditions under which the indirect utility function is increasing in the interest rate and the conditions under which it is decreasing. Interpret these conditions.

Solution: Let $w = (w_1, w_2)$. Using the demand formula for Cobb-Douglas utility functions, we obtain

$$x_1(r,w) = \frac{\alpha\big((1+r)w_1 + w_2\big)}{(1+r)} \quad \text{and} \quad x_2(r,w) = \frac{\beta\big((1+r)w_1 + w_2\big)}{1}$$

$$\implies v(r,w) = \left(\frac{\alpha\big((1+r)w_1 + w_2\big)}{(1+r)}\right)^{\alpha}\left(\frac{\beta\big((1+r)w_1 + w_2\big)}{1}\right)^{\beta} = \frac{\alpha^{\alpha}\beta^{\beta}\big((1+r)w_1 + w_2\big)}{(1+r)^{\alpha}}$$

$$\implies \frac{\partial v(r,w)}{\partial r} = \alpha^{\alpha}\beta^{\beta}\left(\frac{w_1(1+r)^{\alpha} - \big((1+r)w_1 + w_2\big)\alpha(1+r)^{\alpha-1}}{(1+\alpha)^{2\alpha}}\right).$$

Thus, $\frac{\partial v}{\partial r} > 0$ if and only if

$$w_1(1+r)^{\alpha} > \big((1+r)w_1 + w_2\big)\alpha(1+r)^{\alpha-1} \iff (1+r)w_1 > \alpha\big((1+r)w_1 + w_2\big).$$

Note that the right-hand side of the above inequality is the amount of the lifetime income (in future value) that the individual wants to spend on period 1 while the left-hand side is the income in period 1 (in future value). So if the inequality holds, then the individual has more income in period 1 than she wants to spend, so she will be a saver and be better off if the interest rate rises. If the inequality is reversed, then the individual will be a borrower and be worse off if the interest rate rises.

(c) Let $w_1 = 10$ and $w_2 = 100$. Suppose in period 1 the central bank wants to raise the interest rate from $r^0 = 0.2$ to $r^1 = 0.6$. Find the corresponding compensating variation measure of welfare change (CV). (Assume that the compensation is paid in period 2). What happens to the CV as α increases? Interpret this result.

Solution: Using $v(r^1, w - CV) = v(r^0, w)$

$$\frac{\alpha^{\alpha}\beta^{\beta}\big((1+r^1)w_1 + (w_2 - CV)\big)}{(1+r^1)^{\alpha}} = \frac{\alpha^{\alpha}\beta^{\beta}\big((1+r^0)w_1 + w_2\big)}{(1+r^0)^{\alpha}}$$

$$\iff \frac{(1+0.6)(10) + (100 - CV)}{(1+0.6)^{\alpha}} = \frac{\big((1+0.2)(10) + 100\big)}{(1+0.2)^{\alpha}} \iff \frac{116 - CV}{(1.6)^{\alpha}} = \frac{112}{(1.2)^{\alpha}}$$

$$\iff CV = 116 - 112\left(\frac{4}{3}\right)^{\alpha}, \text{ which is decreasing in } \alpha.$$

As α increases, consumption in period 1 becomes more important to the individual, so her savings decrease (or equivalently, borrowing increases), so the benefit of the interest rate increase is reduced.