Advanced Microeconomics I

Fall 2023 - M. Pak

Problem Set 3: due in Class, Nov. 13

1. There are I individuals, and individual i's utility function is given by

$$u_i(x_1,x_2) = a_i x_1 + b_i x_2^{c_i}, \quad \text{where } a_i, b_i > 0 \text{ and } 0 < c_i < 1.$$

   (a) Show that the individuals' indirect utility functions have the Gorman form. You may assume that the solution to the utility maximization problem will be interior and that the second order condition is satisfied.

   Solution: Solving the first order condition yields,

$$MRS = \frac{a_i}{b_i c_i x_2^{c_i-1}} = \frac{p_1}{p_2} \iff x_2 = \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}}.$$

   Substituting this into the budget constraint $p_1 x_1 + p_2 x_2 = w_i$ yields,

$$p_1 x_1 + p_2 \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}} = w_i \implies x_1 = \frac{w_i - p_2 \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}}}{p_1}.$$

   Thus, the indirect utility function is

$$\begin{aligned} v(p,w_i) &= a_i \left(\frac{w_i - p_2 \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}}}{p_1} \right) + b_i \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{c_i}{1-c_i}} \\ &= b_i \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{c_i}{1-c_i}} - \frac{a_i p_2}{p_1} \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}} + \frac{a_i w_i}{p_1}, \end{aligned}$$

   which is a linear function of w_i and thus in Gorman form.

   (b) For what values of a_i, b_i, and c_i will the aggregate demand be a function of price and aggregate wealth only?

   Solution: For the aggregate demand function to depend only on prices and the aggregate wealth, and not the distribution of the aggregate wealth, we need every individual to have an indirect utility function that is in Gorman form (linear function of wealth) with the same slope. Note however that this doe snot require that $a_i = a_j$ for i and j. To see this note that we can transform each individual's utility function by dividing by $a_i > 0$, which preserves the individual's preference. Then, individual i's utility function is now

$$\tilde{u}_i(x_1,x_2) = \frac{u_i(x_1,x_2)}{a_i} = x_1 + \frac{b_i}{a_o}x_2^{c_i}, \quad \text{where } a_i, b_i > 0 \text{ and } 0 < c_i < 1.$$

The corresponding indirect utility function is

$$\tilde{v}(p,w_i) = \frac{v(p,w_i)}{a_i} = \frac{1}{a_i}\left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{c_i}{1-c_i}} - \frac{p_2}{p_1}\left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}} + \frac{w_i}{p_1}.$$

Thus, everyone has the same slope $\frac{1}{p_1}$.

For the remainder of the question, assume that there are only two consumers ($I = 2$), and $a_1 = 1$, $a_2 = 2$, $b_1 = b_2 = 1$, and $c_1 = c_2 = \frac{1}{2}$. Suppose the social welfare function for the economy is given by $W(u_1,u_2) = \min\{u_1,u_2\}$.

(c) Determine the circumstances under which there will be a representative consumer who can be used for aggregate welfare analysis.

Solution: A representative consumer exists when wealth assignment maximizes the Social welfare function. That is, if it solves:

$$\max_{w_1,w_2} W(v_1(p,w_1), v_2(p,w_2)) \text{ s.t. } w_1+w_2 = w. \iff \max_{w_1} \{(v_1(p,w_1),\ v_2(p,w-w_1)\}.$$

From part (a), we have,

$$v_1(p,w_1) = \frac{p_1}{2p_2} - \frac{p_2}{p_1}\left(\frac{p_1}{2p_2}\right)^2 + \frac{w_1}{p_1} = \frac{p_1}{2p_2} - \frac{p_1}{4p_2} + \frac{w_1}{p_1} = \frac{p_1}{4p_2} + \frac{w_1}{p_1}$$
$$v_2(p,w_2) = \frac{p_1}{4p_2} - \frac{2p_2}{p_1}\left(\frac{p_1}{4p_2}\right)^2 + \frac{2w_2}{p_1} = \frac{p_1}{4p_2} - \frac{p_1}{8p_2} + \frac{2w_2}{p_1} = \frac{p_1}{8p_2} + \frac{2w_2}{p_1}.$$

The solution to the maximization occurs where $v_1 = v_2$. Thus,

$$\frac{p_1}{4p_2} + \frac{w_1}{p_1} = \frac{p_1}{8p_2} + \frac{2(w-w_1)}{p_1} \iff \frac{p_1^2}{4p_2} + w_1 = \frac{p_1^2}{8p_2} + 2w - 2w_2$$
$$\iff w_1^* = \frac{2w}{3} - \frac{p_1^2}{24p_2} \implies w_2^* = \frac{w}{3} + \frac{p_1^2}{24p_2}$$

(d) Find the indirect utility function for the representative consumer.

Solution:

$$V(p,w) = \min\{v_1(p,w_1^*),\ v_2(p_2,w_2^*)\} = \frac{p_1}{4p_2} + \frac{1}{p_1}\left(\frac{2w}{3} - \frac{p_1^2}{24p_2}\right)$$
$$= \frac{2w}{3p_1} + \frac{p_1}{4p_2} - \frac{p_1}{24p_2} = \frac{5}{24p_2} + \frac{2w}{3p_1}.$$

2. A firm uses a production function $f(z) = \min\ \{z_1^a,\ z_2^b\}$, $z \in \mathbb{R}_+^L$, to produce an output.

(a) Find the firm's cost function. Determine the conditions under which the cost function will be concave, linear, and convex function of the output level, y, respectively (you may use the second derivative condition).

Solution: For Leontief production function, cost-minimizing bundle occurs at the corner point: $z_1^a = z_2^b \implies z_2 = z_1^{\frac{a}{b}}$. Thus,

$$\min\left\{z_1^a,\ \left(z_1^{\frac{a}{b}}\right)^b\right\} = y \implies z_1(w,y) = y^{\frac{1}{a}} \implies z_2(w,y) = y^{\frac{1}{b}} \implies c(w,y) = w_1 y^{\frac{1}{a}} + w_2 y^{\frac{1}{b}}$$

$$\implies \frac{\partial^2 c(w,y)}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{w_1}{a} y^{\frac{1}{a}-1} + \frac{w_2}{b} y^{\frac{1}{b}-1}\right) = \frac{w_1}{a}\left(\frac{1}{a}-1\right) y^{\frac{1}{a}-2} + \frac{w_2}{b}\left(\frac{1}{b}-1\right) y^{\frac{1}{b}-2}.$$

Since the second derivative condition for concavity/convexity has to hold for all $w_1 > 0$ and $w_2 > 0$, we have

$$c(w,y) \text{ is } \begin{cases} \text{concave} & \text{if } \frac{1}{a} \le 1 \text{ and } \frac{1}{b} \le 1, \text{ or equivalently } a \ge 1 \text{ and } b \ge 1 \\ \text{linear} & \text{if } a = b = 1 \\ \text{convex} & \text{if } a \le 1 \text{ and } b \le 1. \end{cases}$$

(b) Assuming that the cost function is convex and that $a = b$, find the firm's profit function.

Solution:

$$p = c'(w,y) \implies p = \frac{w_1}{a} y^{\frac{1}{a}-1} + \frac{w_2}{a} y^{\frac{1}{a}-1} \implies p = \frac{w_1 + w_2}{a} y^{\frac{1-a}{a}} \implies y = \left(\frac{ap}{w_1 + w_2}\right)^{\frac{a}{1-a}}.$$

$$\pi(p,w) = p\left(\frac{ap}{w_1 + w_2}\right)^{\frac{a}{1-a}} - (w_1 + w_2)\left(\frac{ap}{w_1 + w_2}\right)^{\frac{1}{1-a}}$$

(c) Assuming that the cost function is linear and that $a = b$, find the firm's profit function.

Solution: Linear cost function means that $a = b = 1$. Thus,

$$\text{marginal profit } = p - c'(w,y) = p - (w_1 + w_2).$$

Therefore,

$$y(p,w) = \begin{cases} 0 & \text{if } p < w_1 + w_2 \\ [0, \infty) & \text{if } p = w_1 + w_2 \\ \text{undefined} & \text{if } p > w_1 + w_2 \end{cases}$$

$$\implies \pi(p,w) = \begin{cases} 0 & \text{if } p \le w_1 + w_2 \\ \text{undefined} & \text{if } p > w_1 + w_2. \end{cases}$$

3. Suppose a firm's (single-output) production function is $f(z) = z_1^b + z_2^b$, where $b > 0$. In the following, let $q > 0$ denote the output quantity.

(a) Find all the values of b for which the firm's production function is quasiconcave (that is, its upper contour sets, $\{z \in \mathbb{R}^2_+ : f(z) \geq q\}$, are convex sets).

Solution: Let $z = (z_1, z_2)$ and $z' = (z'_1, z'_2)$ be such that $f(z) = f(z')$. Let $z'' = \alpha z + (1-\alpha)z'$ for some $0 < \alpha < 1$. First, note that $g(z_\ell) = z_\ell^b$ is concave if and only if $0 \leq b \leq 1$. That is, $\left(\alpha z_\ell + (1-\alpha)z'_\ell\right)^b > \alpha z_\ell^b + (1-\alpha)z_\ell'^b$ if and only if $0 \leq b \leq 1$. Next, note that we need $f(z'') \geq f(z)$ for quasiconcavity. This means,

$$\begin{aligned} f(z'') &= \left(\alpha z_1 + (1-\alpha)z'_1\right)^b + \left(\alpha z_2 + (1-\alpha)z'_2\right)^b \\ &\geq \alpha z_1^b + (1-\alpha)z_1'^b + \alpha z_2^b + (1-\alpha)z_2'^b \quad \text{if and only if } 0 \leq b \leq 1 \\ &= \alpha f(z) + (1-\alpha)f(z') = \alpha f(z) + (1-\alpha)f(z) = f(z). \end{aligned}$$

Therefore, we need $0 \leq b \leq 1$.

(b) Find all the values of b for which the firm's production is strictly concave.

Solution: We have $\frac{\partial f}{\partial z_\ell} = b z_\ell^{b-1}$. Thus, for all $x \neq 0$,

$$\begin{aligned} x^T H_f x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} b(b-1)z_1^{b-2} & 0 \\ 0 & b(b-1)z_1^{b-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= b(b-1)x_1^2 z_1^{b-2} + b(b-1)x_1^2 z_2^{b-2} < 0 \iff 0 < b < 1. \end{aligned}$$

(c) Assuming that the firm's production function is strictly concave, find the firm's cost function and show that it is a strictly convex function of the output level q.

Solution:

$$\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} = \frac{b z_1^{b-1}}{b z_2^{b-1}} = \frac{w_1}{w_2} \iff \frac{z_2^{1-b}}{z_1^{1-b}} = \frac{w_1}{w_2} \iff z_2 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{1-b}} z_1.$$

$$z_1^b + \left(\frac{w_1}{w_2}\right)^{\frac{b}{1-b}} z_1^b = q \implies z_1^b = \left(\frac{w_1^{\frac{b}{1-b}} + w_2^{\frac{b}{1-b}}}{w_2^{\frac{b}{1-b}}}\right)^{-1} q \implies z_1 = \left(\frac{w_2^{\frac{b}{1-b}}}{w_1^{\frac{b}{1-b}} + w_2^{\frac{b}{1-b}}}\right)^{\frac{1}{b}} q^{\frac{1}{b}}.$$

$$c(w,q) = w_1 \left(\frac{w_2^{\frac{b}{1-b}}}{w_1^{\frac{b}{1-b}} + w_2^{\frac{b}{1-b}}}\right)^{\frac{1}{b}} q^{\frac{1}{b}} + w_2 \left(\frac{w_1^{\frac{b}{1-b}}}{w_1^{\frac{b}{1-b}} + w_2^{\frac{b}{1-b}}}\right)^{\frac{1}{b}} q^{\frac{1}{b}},$$

which is strictly concave since $\frac{1}{b} > 1$.