Advanced Microeconomics I

Fall 2024 - M. Pak

Midterm: Suggested Solutions

1. [25] Consider the indirect utility function $v(p,w) = \ln\left(\frac{w^2}{4p_1p_2}\right)$.

   (a) [6] Find the Marshallian demand.
   Solution: We have $v(p,w) = \ln w^2 - \ln 4 - \ln p_1 - \ln p_2$. Therefore,

   $$\forall \ell,\ x_\ell(p,w) = -\frac{\frac{\partial v}{\partial p_\ell}}{\frac{\partial v}{\partial w}} = -\frac{-\frac{1}{p_\ell}}{\frac{2w}{w^2}} = \frac{w}{2p_\ell}.$$

   (b) [6] Find the expenditure function and the Hicksian demand.
   Solution: Using $v(p,e(p,u)) = u$, yields

   $$\ln\left(\frac{e(p,u)^2}{4p_1p_2}\right) = u \implies e(p,u)^2 = 4p_1p_2e^u \implies e(p,u) = 2\left(p_1p_2e^u\right)^{\frac{1}{2}} = 2p_1^{\frac{1}{2}}p_2^{\frac{1}{2}}e^{\frac{u}{2}}$$

   $$h_1(p,u) = \frac{\partial e(p,u)}{\partial p_\ell} = p_1^{-\frac{1}{2}}p_2^{\frac{1}{2}}e^{\frac{u}{2}} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{2}}e^{\frac{u}{2}} \implies h_2(p,u) = \left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}e^{\frac{u}{2}}.$$

   (c) [6] Suppose currently $w = 100$, $p_1 = 1$, and $p_2 = 4$. What is the income effect on good 2 when the price of good 1 changes. Is good 2 a normal good or an inferior good?
   Solution:

   $$-x_1(p,w)\frac{\partial x_2(p,w)}{\partial w} = -\left(\frac{w}{2p_1}\right)\left(\frac{1}{2p_2}\right) = -\frac{100}{2(1)(2)(4)} = -\frac{25}{4} \implies \text{normal good.}$$

   (d) [7] State the Slutksy equation for good 2 when the price of good 1 changes. Verify by explicit calculation that it holds for the case in part (c).
   Solution: $u^o = v(p,w) = \ln\left(\frac{w^2}{4p_1p_2}\right) = \ln\left(\frac{100^2}{4(1)(4)}\right) = \ln\ 625$. Thus,

   $$\frac{\partial h_2}{\partial p_1} = \frac{1}{2}p_1^{-\frac{1}{2}}p_2^{-\frac{1}{2}}e^{\frac{u}{2}} = \frac{e^{\frac{\ln 625}{2}}}{2(1)(2)} = \frac{625^{\frac{1}{2}}}{4} = \frac{25}{4} = 6.25.$$

   $$\frac{\partial x_2}{\partial p_1} = 0 \quad \text{and} \quad x_1\frac{\partial x_2}{\partial w} = +\frac{25}{4} \text{ from part (c).}$$

   $$\implies \frac{\partial h_2}{\partial p_1} = \frac{25}{4} = 0 + \frac{25}{4} = \frac{\partial x_2}{\partial p_1} + x_1\frac{\partial x_2}{\partial w}, \quad \text{as required.}$$

2. [25] Bob's utility function is $u(x) = x_1 + 2x_2^{\frac{1}{2}}$. In the following, assume that demand functions will be interior.

   (a) [7] Show that Bob's indirect utility function has a Gorman form.

   Solution: Assuming interior solution, we have

$$MRS = \frac{1}{x_2^{-\frac{1}{2}}} = x_2^{\frac{1}{2}} = \frac{p_1}{p_2} \implies x_2^* = \left(\frac{p_1}{p_2}\right)^2 \implies x_1^* = \frac{w - p_2\left(\frac{p_1}{p_2}\right)^2}{p_1} = \frac{w}{p_1} - \frac{p_1}{p_2}$$

$$\implies v(p,w) = \left(\frac{w}{p_1} - \frac{p_1}{p_2}\right) + 2\left(\left(\frac{p_1}{p_2}\right)^2\right)^{\frac{1}{2}} = \frac{w}{p_1} - \frac{p_1}{p_2} + \frac{2p_1}{p_2} = \underbrace{\frac{p_1}{p_2}}_{a(p)} + \underbrace{\frac{1}{p_1}}_{b(p)} w.$$

   So, the indirect utility is a linear function of w, as required.

   (b) [7] Currently, $p_1 = 20$ RMB, $p_2 = 2$ RMB, and Bob's wealth is $w = 200$ RMB. Suppose the government is thinking about imposing 2 RMB per-unit tax on good 2, which means Bob must pay $p_2 + 2 = 4$ RMB for each unit of good 2. How much tax revenue will the government raise from Bob?

   solution: Letting T be the tax revenue, we have

$$T = t \times x_2(p,t,w) = t \times \left(\frac{p_1}{p_2 + t}\right)^2 = 2\left(\frac{20}{4}\right)^2 = 50 \text{ RMB}.$$

   (c) [7] In general (that is, not just for Bob), should equivalent variation or compensating variation measure of welfare be used to find the maximum amount of money an individual is willing to pay to avoid this tax? Find Bob's maximum willingness to pay using the correct general method.

   Solution: EV should be used. Using $v(p, w + EV) = v(p', w)$, we obtain

$$\frac{20}{2} + \frac{200 + EV}{20} = \frac{20}{4} + \frac{200}{20} \iff \frac{200 + EV}{20} = 5 + 10 - 10$$
$$\implies EV = 20(5) - 200 = -100 \text{ RMB}.$$

   So Bob is willing to pay up to 100 RMB to avoid the tax.

   (d) [2] Will Bob be better off if the government imposed the tax, or if the government directly asked Bob to pay the amount found in part (b)?

   Solution: Since $|EV| = 100 > 50 = T$, Bob will be better off if he just paid 50 RMB to the government. (This is an example of tax leading to a dead-weight loss.)

   (e) [2] For the proposed tax, would Bob's equivalent variation measure of welfare change be larger, equal, or smaller than compensating variation measure of welfare change for Bob?

   Solution: As seen in part (s). The wealth effect is zero on good 2. Therefore, EV and CV will be the same.

3. [20] An economy consists of 2 individuals, whose utility functions are $u_1(x_{11}, x_{21}) = \min\{2x_{11}, x_{21}\}$ and $u_2(x_{12}, x_{22}) = \min\{x_{12}, 2x_{22}\}$ and wealths are w_1 and $w2$, respectively.

   (a) [10] Find the individuals' indirect utility functions. Show whether the aggregate demand, as a function of prices and aggregate wealth, exists.

   Solution:

$$2x_{11} = x_{21} \implies p_1 x_{11} + p_2(2x_{11}) = w_1 \implies x_{11}^* = \frac{w_1}{p_1 + 2p_2} \implies x_{21}^* = \frac{2w_1}{p_1 + 2p_2}$$

$$x_{12} = 2x_{22} \implies p_1 x_{12} + p_2(\tfrac{1}{2}x_{12}) = w_2 \implies x_{12}^* = \frac{2w_2}{2p_1 + p_2} \implies x_{22}^* = \frac{w_2}{2p_1 + p_2}$$

$$\implies v_1(p, w) = \underbrace{\left(\frac{2}{p_1 + 2p_2}\right)}_{b_1(p)} w_1 \quad \text{and} \quad v_2(p, w) = \underbrace{\left(\frac{2}{2p_1 + p_2}\right)}_{b_2(p)} w_2.$$

   Although both indirect utilities are in Gorman form, $b_1(p) \neq b_2(p)$, and there is no increasing transformation of utilities that will make them equal. Thus, aggregate demand as a function of aggregate wealth does not exist.

   (b) [10] Suppose a central planner wishes to maximize social welfare function $W(u_1, u_2) = \min\{u_1, u_2\}$, where u_i is individual i's utility. What is the optimal wealth allocation rule and the resulting indirect utility function of the representative individual?

   Solution: Using $v_1(p, w_1) = v_2(p, w_2)$ and $w_1 + w_2 = w$ yields,

$$\left(\frac{2}{p_1 + 2p_2}\right) w_1 = \left(\frac{2}{2p_1 + p_2}\right)(w - w_1) \iff (2p_1 + P_2)w_1 = (p_1 + 2p_2)w - (p_1 + 2p_2)w_1$$

$$\iff (3p_1 + 3P_2)w_1 = (p_1 + 2p_2)w \implies w_1^* = \frac{p_1 + 2p_2}{3p_1 + 3p_2} \implies w_2^* = \left(\frac{2p_1 + p_2}{3p_1 + 3p_2}\right) w$$

$$\implies V(p, w) = v_1(p, w_1^*) = \left(\frac{2}{p_1 + 2p_2}\right)\left(\frac{p_1 + 2p_2}{3p_1 + 3p_2}\right) w = \frac{2w}{3p_1 + 3p_2}$$

4. [15] Consider the production function $f(z) = z_1^{\frac{1}{4}} z_2^{\frac{1}{4}}$.

(a) [8] Find the conditional input demand and the cost function.

Solution:

$$\text{MRTS} = \frac{\frac{1}{4} z_1^{-\frac{3}{4}} z_2^{\frac{1}{4}}}{\frac{1}{4} z_1^{\frac{1}{4}} z_2^{-\frac{3}{4}}} = \frac{z_2}{z_1} = \frac{w_1}{w_2} \implies z_2 = \frac{w_1}{w_2} z_1$$

$$z_1^{\frac{1}{4}} \left(\frac{w_1}{w_2} z_1\right)^{\frac{1}{4}} = q \implies z_1^{\frac{1}{2}} = \left(\frac{w_2}{w_1}\right)^{\frac{1}{4}} q \implies z_1(w,q) = \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}} q^2 \implies z_2(w,q) = \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}} q^2$$

$$c(w,q) = w_1 \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}} q^2 + w_2 \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}} q^2 = 2(w_1 w_2)^{\frac{1}{2}} q^2.$$

(b) [7] Find the supply function and the unconditional input demand.

Solution: Since cost function is strictly convex, the first order condition $p = MC$ is necessary and sufficient for profit maximization.

$$p = 4(w_1 w_2)^{\frac{1}{2}} q \implies y(p,w) = \frac{p}{4(w_1 w_2)^{\frac{1}{2}}}$$

$$z_1(p,w) = z_1\big(w, y(p,w)\big) = \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}} \left(\frac{p}{4(w_1 w_2)^{\frac{1}{2}}}\right)^2 = \frac{p^2}{16 w_1^{\frac{3}{2}} w_2^{\frac{1}{2}}}$$

$$z_2(p,w) = \frac{p^2}{16 w_1^{\frac{1}{2}} w_2^{\frac{3}{2}}} \quad \text{by symmetry.}$$

5. [15] Consider a firm with production function $f(z)$, $z \in \mathbb{R}^L_+$ with strictly convex isoquants. Let p be the output price and w be the input prices.

(a) [6] Show whether the profit function $\pi(p,w)$ is a convex or concave function of prices.

Solution: Choose any (p,w), (p',w'), and $\alpha \in [0,1]$, and let $(p'',w'') = \alpha(p,w)+(1-\alpha)(p',w')$. Since $z(p'',w'')$ is not necessarily the profit maximizing input choice when prices are (p,w) or (p',w'), we have

$$\alpha\left[pf\left(z(p'',w'')\right) - w\cdot z(p'',w'')\right] \le \alpha\left[\pi(p,w)\right]$$
$$(1-\alpha)\left[p'f\left(z(p'',w'')\right) - w'\cdot z(p'',w'')\right] \le (1-\alpha)\left[\pi(p',w')\right].$$

Therefore,

$$\begin{aligned}\pi(p'',w'') &= p''f\left(z(p'',w'')\right) - w''\cdot\left(z(p'',w'')\right)\\ &= \left(\alpha p + (1-\alpha)p'\right)f\left(z(p'',w'')\right) - \left(\alpha w + (1-\alpha)w'\right)\cdot z(p'',w'')\\ &\le \alpha\pi(p,w) + (1-\alpha)\pi(p',w').\end{aligned}$$

Therefore, $\pi(\cdot)$ is convex, as required.

(b) [4] Suppose the firm's profit function is known. How can the supply function and the unconditional input demand be derived from this knowledge?

Solution: By Hotelling's lemma,

$$y(p,w) = \frac{\partial \pi}{\partial p} \quad \text{and} \quad z_\ell(p,w) = -\frac{\partial \pi}{\partial w_\ell}\ \forall \ell.$$

(c) [5] Suppose a government imposes a value-added tax of t fraction on the price of the output good, so the firm only receives $(1-t)p$ for each unit it sells. Suppose the firm's profit function $\pi(p,w,t)$ is known. Show whether the supply function and the unconditional input demand can be derived from this knowledge.

Solution: Now the firm's profit maximization problem is

$$\max_z\ (1-p)tf(z) - w\cdot z.$$

Since tax does not affect the cost, we still have $z_\ell(p,w) = -\frac{\partial\pi(p,w,t)}{\partial w_\ell}$ for all ℓ. For the supply function, we have by envelope theorem,

$$\frac{\partial \pi(p,w,t)}{\partial p} = -tf(z^*) \implies y(p,w,t) = f(z^*) = -\frac{\partial^2\pi(p,w,t)}{\partial t\partial p}.$$

Or, alternatively, we can use

$$y(p,w,t) = -\frac{\frac{\partial\pi(p,w,t)}{\partial p}}{t} \quad \text{or} \quad y(p,w,t) = \frac{\frac{\partial\pi(p,w,t)}{\partial t}}{1-p}.$$