Advanced Microeconomics I

Fall 2024 - M. Pak

Exercises: Producer Theory

1 Producer Theory

1. Suppose the production set $Y \subset \mathbb{R}^n$ is closed and satisfies the free disposal property. Let $\pi(p)$, be the corresponding profit function. Show that $\pi(p)$, $p \in \mathbb{R}^n_{++}$, is convex in p.

   Solution: Take any $\alpha \in [0,1]$. Then

$$\begin{aligned} \alpha p \cdot y(\alpha p + (1-\alpha)p') &\leq \alpha p \cdot y(p), \quad \text{and} \\ (1-\alpha)p' \cdot y(\alpha p + (1-\alpha)p') &\leq (1-\alpha)p' \cdot y(p') \\ \Rightarrow \left(\alpha p + (1-\alpha)p'\right) \cdot y(\alpha p + (1-\alpha)p') &\leq \alpha p \cdot y(p) + (1-\alpha)p' \cdot y(p') \\ \Rightarrow \pi(\alpha p + (1-\alpha)p') &\leq \alpha \pi(p) + (1-\alpha)\pi(p'). \end{aligned}$$

2. Suppose a firm's production function is given by

$$f(z_1, z_2) = \min\{z_1, z_2\}.$$

   (a) Find the firm's cost function.

   Solution: The solution to the cost-minimization problem

$$\min_{z_1,z_2} w_1 z_1 + w_2 z_2 \quad \text{s.t.} \quad \min\{z_1, z_2\} = y$$

   is characterized by two conditions:

$$z_1 = z_2 \quad \text{and} \quad \min\{z_1, z_2\} = y$$

   Therefore, $z_1(w,y) = z_2(w,y) = y$, and the cost function is $c(w,y) = (w_1 + w_2)y$

   (b) Is the production function differentiable? Is the cost function differentiable? On the scale of 0 to 10 (0 = "not at all interesting"; 10 = "I don't know how I am going to sleep tonight"), rate how interesting you find this fact.

   Solution: Notice that both the conditional demand function and the cost function is differentiable even though the production function is not. Should we not find the fact that solutions and value functions can be differentiable even though the objective function or the constraint function are not, at least somewhat interesting?

3. Consider the production function $f:\mathbb{R}^2_+ \to \mathbb{R}_+$ given by

$$f(z) = z_1^{\frac{1}{3}} z_2^{\frac{1}{3}}.$$

(a) Find the firm's conditional input demand function and the cost function.
Solution: cost minimization problem is given by:

$$\min_z \; w\cdot z \quad \text{s.t.} \quad f(z) = y,$$

whose associated Lagrangian is

$$\mathscr{L} = w\cdot z + \mu\left[y - z_1^{\frac{1}{3}} z_2^{\frac{1}{3}}\right].$$

The first order condition is given by

$$\frac{z_1^{-\frac{2}{3}} z_2^{\frac{1}{3}}}{3} = w_1 \quad \text{and} \quad \frac{z_1^{\frac{1}{3}} z_2^{-\frac{2}{3}}}{3} = w_2 \quad \Rightarrow \quad \frac{z_2}{z_1} = \frac{w_1}{w_2}.$$

Substituting this into the production function $z_1^{\frac{1}{3}} z_2^{\frac{1}{3}} = y$ yields

$$z_1^{\frac{1}{3}}\left(\frac{w_1}{w_2} z_1\right)^{\frac{1}{3}} = y \quad \Rightarrow \quad z_1^2 \frac{w_1}{w_2} = y^3 \quad \Rightarrow \quad z_1(w,y) = \left(\frac{w_2 y^3}{w_1}\right)^{\frac{1}{2}},$$

$$\text{and} \quad z_2(w,y) = \left(\frac{w_1 y^3}{w_2}\right)^{\frac{1}{2}}.$$

The cost function is given by

$$c(w,y) = w_1\left(\frac{w_2 y^3}{w_1}\right)^{\frac{1}{2}} + w_2\left(\frac{w_1 y^3}{w_2}\right)^{\frac{1}{2}} = 2\left(w_1 w_2 y^3\right)^{\frac{1}{2}}$$

(b) Find the firm's supply function and the profit function by explicitly solving $\max_y py - c(p,y)$
Solution: The profit maximization problem is

$$\max_y \; py - 2\left(w_1 w_2 y^3\right)^{\frac{1}{2}}.$$

Solving the first order condition yields

$$p = 3(w_1 w_2)^{\frac{1}{2}} y^{\frac{1}{2}} \quad \Rightarrow \quad y(p,w) = \frac{p^2}{9 w_1 w_2},$$

$$\pi(p,w) = \frac{p^3}{9 w_1 w_2} - 2\left(w_1 w_2 y^3\right)^{\frac{1}{2}}.$$

(c) Find the firm's unconditional input demand function. Verify that $z(p,w) = z(p,y(p,w))$.

Solution: The profit maximization is now formulated as

$$\max_z \; p\left(z_1^{\frac{1}{3}} z_2^{\frac{1}{3}}\right) - w \cdot z.$$

$$\frac{p z_1^{-\frac{2}{3}} z_2^{\frac{1}{3}}}{3} = w_1 \quad \text{and} \quad \frac{p z_1^{\frac{1}{3}} z_2^{-\frac{2}{3}}}{3} = w_2 \qquad \Rightarrow \qquad \frac{z_2}{z_1} = \frac{w_1}{w_2}.$$

Substituting this back into the first equation yields

$$\frac{p z_1^{-\frac{2}{3}} \left(\frac{w_1 z_1}{w_2}\right)^{\frac{1}{3}}}{3} = w_1 \quad \Rightarrow \quad \frac{p^3 z_1^{-1} \frac{w_1}{w_2}}{27} = w_1^3 \quad \Rightarrow \quad z_1(p,w) = \frac{p^3}{27 w_1^2 w_2}$$

$$\Rightarrow \quad z_2(p,w) = \frac{p^3}{27 w_1 w_2^2}.$$

Therefore, we have

$$z_1(p,y(p,w)) = \left(\frac{w_2 \left(\frac{p^2}{9 w_1 w_2}\right)^3}{w_1}\right)^{\frac{1}{2}} = \frac{p^3}{27 w_1^2 w_2}$$

$$z_2(p,y(p,w)) = \left(\frac{w_1 \left(\frac{p^2}{9 w_1 w_2}\right)^3}{w_2}\right)^{\frac{1}{2}} = \frac{p^3}{27 w_1 w_2^2}.$$

(d) Suppose the firm's production function is given instead by

$$\tilde{f}(z) = [f(z)]^{\frac{3}{2}} = z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}$$

Find the firm's profit function.

Solution: The first order condition for the cost minimization problem is given by

$$\frac{z_1^{-\frac{1}{2}} z_2^{\frac{1}{2}}}{2} = w_1 \quad \text{and} \quad \frac{z_1^{\frac{1}{2}} z_2^{-\frac{1}{2}}}{2} = w_2 \qquad \Rightarrow \qquad z_2 = \frac{w_1 z_1}{w_2}.$$

Substituting this into the production function $z_1^{\frac{1}{2}} z_2^{\frac{1}{2}} = y$ yields

$$z_1^{\frac{1}{2}} \left(\frac{w_1 z_1}{w_2}\right)^{\frac{1}{2}} = y \quad \Rightarrow \quad z_1(w,y) = \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}} y,$$

$$\text{and} \quad z_2(w,y) = \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}} y.$$

The cost function is given by

$$c(w,y) = w_1 \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}} y + w_2 \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}} y = 2 (w_1 w_2)^{\frac{1}{2}} y$$

The profit maximization problem is

$$\max_{y} \; py - 2(w_1 w_2)^{\frac{1}{2}} y.$$

Solving the first order condition yields

$$p = 2(w_1 w_2)^{\frac{1}{2}}.$$

So, an interior solution exists if and only if $p = 2(w_1 w_2)^{\frac{1}{2}}$. In fact,

$$y(p,w) = \begin{cases} 0 & \text{if } p < 2(w_1 w_2)^{\frac{1}{2}} \\ \text{any } y & \text{if } p = 2(w_1 w_2)^{\frac{1}{2}} \\ \infty & \text{if } p > 2(w_1 w_2)^{\frac{1}{2}} \end{cases},$$

and the profit function is

$$\pi(p,w) = \begin{cases} 0 & \text{if } p < 2(w_1 w_2)^{\frac{1}{2}} \\ 0 & \text{if } p = 2(w_1 w_2)^{\frac{1}{2}} \\ \infty & \text{if } p > 2(w_1 w_2)^{\frac{1}{2}} \end{cases}.$$

Note that with CRS production function, an interior solution exists only if the firm is making zero profit.

4. For the profit function

$$\pi(p,w) = \frac{p^2}{8 w_1^{\frac{1}{2}} w_2^{\frac{1}{2}}},$$

(a) Find the unconditional input demands.

Solution: The profit function $\pi(p,w)$ is the value function for the profit maximization problem:

$$\max_{x} \; pf(x) - w \cdot x.$$

Therefore, the envelope theorem yields

$$\begin{aligned} x_i(p,w) &= -\frac{\partial \pi(p,w)}{\partial w_i} \quad \text{(Hotelling's Lemma)} \\ \Rightarrow x_1(p,w) &= \frac{p^2}{16 w_1^{\frac{3}{2}} w_2^{\frac{1}{2}}}, \quad \text{and} \\ \Rightarrow x_2(p,w) &= \frac{p^2}{16 w_1^{\frac{1}{2}} w_2^{\frac{3}{2}}}. \end{aligned}$$

(b) Find the supply function.

Solution: The envelope theorem again yields

$$\begin{aligned} y(p,w) = f(x(p,w)) &= \frac{\partial \pi(p,w)}{\partial p} \\ &= \frac{p}{4 w_1^{\frac{1}{2}} w_2^{\frac{1}{2}}}. \end{aligned}$$

(c) Find the cost function.

Solution: From part(b), we have

$$p = 4yw_1^{\frac{1}{2}}w_2^{\frac{1}{2}}$$

Using the first order condition $p = MC$, we obtain

$$\begin{aligned}
\frac{\partial c(w,y)}{\partial y} &= 4yw_1^{\frac{1}{2}}w_2^{\frac{1}{2}} \\
\Rightarrow c(w,y) &= \int^y 4zw_1^{\frac{1}{2}}w_2^{\frac{1}{2}}\,dz \\
&= 2y^2w_1^{\frac{1}{2}}w_2^{\frac{1}{2}} + K,
\end{aligned}$$

where K is a constant. (Though not required, value of K can be determined after the production function is found in part (e)).

(d) Find the conditional input demands.

Solution: The cost function $c(w,y)$ is the value function for the cost minimization problem:

$$\min_x \; w \cdot x \quad \text{s.t.} \quad f(x) = y,$$

whose associated Lagrangian is

$$\mathscr{L} = w \cdot x + \mu[y - f(x)].$$

Therefore, the envelope theorem yields

$$\begin{aligned}
x_i(w,y) &= \frac{\partial c(w,y)}{\partial w_i} \quad \text{(Shepard's Lemma)} \\
\Rightarrow x_1(p,w) &= \frac{y^2w_2^{\frac{1}{2}}}{w_1^{\frac{1}{2}}}, \quad \text{and} \\
\Rightarrow x_2(p,w) &= \frac{y^2w_1^{\frac{1}{2}}}{w_2^{\frac{1}{2}}}.
\end{aligned}$$

(e) Find the production function.

Solution: From part(d), we have

$$\begin{aligned}
x_1w_1^{\frac{1}{2}}w_2^{-\frac{1}{2}} &= y^2 \\
x_2w_1^{-\frac{1}{2}}w_2^{\frac{1}{2}} &= y^2 \\
\Rightarrow \quad x_1x_2 &= y^4 \\
\Rightarrow \quad y &= x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}.
\end{aligned}$$

5. Consider a firm with single-output production function

$$f(z) = \sqrt{\min\{z_1, z_2\}}.$$

(a) Find the firm's supply function.

Solution: We first solve the cost minimization problem:

$$\min\ w\cdot q \quad \text{s.t.} \quad \sqrt{\min\{z_1,z_2\}}=q$$

Since $f(z)$ is a monotonic transformation of a Leontieff production function, the conditional input demands can be found easily:

$$z_1=z_2 \quad\Rightarrow\quad \sqrt{\min\{z_1,z_1\}}=q \quad\Rightarrow\quad z_1(w,q)=z_2(w,q)=q^2.$$

So, the cost function is $c(q)=(w_1+w_2)q^2$.

Next, profit maximization requires:

$$p=MC=2(w_1+w_2)q \quad\Rightarrow\quad y(p,w)=\frac{p}{2(w_1+w_2)}.$$

(b) Find the firm's *unconditional* input demand function.

Solution: Substituting the supply function into the conditional input demand function yields:

$$z_1(p,w)=z_2(p,w)=\left(\frac{p}{2(w_1+w_2)}\right)^2=\frac{p^2}{4(w_1+w_2)^2}.$$

(c) Suppose firm's production function had been $f(z)=(\min\{z_1,z_2\})^2$. What is the firm's supply function?

Solution: Sine the production plan exhibits increasing returns to scale, there is not solution to the profit maximization problem and the supply function does not exist.

6. In the following, you may assume as much differentiability as needed.

(a) Show that the profit function $\pi(p,w)$ is convex in (p,w).

Solution: We need to show that

$$\underbrace{\pi(\alpha(p',w')+(1-\alpha)(p'',w''))}_{LHS} \le \underbrace{\alpha\pi(p',w')+(1-\alpha)\pi(p'',w'')}_{RHS}.$$

$$\begin{aligned} RHS &= \max_{x',x''}\left[\alpha p'f(x')-\alpha w'\cdot x'\right]+\left[(1-\alpha)p''f(x'')-(1-\alpha)w''\cdot x''\right] \\ &\ge \max_{x'}\left[\alpha p'f(x')-\alpha w'\cdot x'\right]+\left[(1-\alpha)p''f(x')-(1-\alpha)w''\cdot x'\right] \\ &\ge \max_{x'}(\alpha p'+(1-\alpha)p'')f(x')-(\alpha w'+(1-\alpha)w'')\cdot x' \\ &= LHS \end{aligned}$$

(b) Does the law of supply hold? That is, is

$$\frac{\partial y(p,w)}{\partial p}\ge 0\,?$$

Solution: Using Hotelling's Lemma,

$$\frac{\partial y(p,w)}{\partial p} = \frac{\partial}{\partial p}\left[\frac{\partial \pi(p,w)}{\partial p}\right] \ge 0 \text{ by convexity of } \pi(\cdot).$$

(c) Do the unconditional input demands satisfy the law of demand? That is, is

$$\frac{\partial x_\ell(p,w)}{\partial p_\ell} \le 0 \text{ for all } \ell \text{ ?}$$

Solution: Using Hotelling's Lemma again,

$$\frac{\partial x_\ell(p,w)}{\partial w_\ell} = -\frac{\partial}{\partial w_\ell}\left[\frac{\partial \pi(p,w)}{\partial w_\ell}\right] \le 0 \text{ by convexity of } \pi(\cdot).$$

7. LetsBurnInvestorsMoney.com is a startup firm that produces a product called *pipedream*. The firm acts as a price taker; however, the firm has decided that growth is more important than profitability. Therefore, the firm has chosen to maximize output instead of profit. What the firm can spend on input goods is constrained by how much cash it has on hand.

   Suppose your careful observation of the firm's behavior has revealed that when the input prices are $w = (w_1, w_2)$ and the firm has \$$C$ cash on hand, the firm's output is:

$$y(w,C) = \left(\frac{1}{3w_1}\right)^{\frac{1}{3}}\left(\frac{2}{3w_2}\right)^{\frac{2}{3}} C.$$

   Find the firm's input demand.

   Solution: This firm solves

$$\max_z f(z) \quad \text{s.t.} \quad w \cdot z = C.$$

   The Lagrangian is

$$\mathscr{L} = f(z) + \lambda[C - w \cdot z].$$

   The value function is

$$y(w,C) = \left(\frac{1}{3w_1}\right)^{\frac{1}{3}}\left(\frac{2}{3w_2}\right)^{\frac{2}{3}} C.$$

   By the envelope theorem (compare with Roy's Identity),

$$\frac{\partial y}{\partial w_i} = \frac{\mathscr{L}}{\partial w_i} = -\lambda z_i \quad \text{and} \quad \frac{\partial y}{\partial C} = \frac{\mathscr{L}}{\partial C} = \lambda.$$

   Therefore, the firm's input demands are

$$z_1(w,C) = -\frac{\frac{\partial y}{\partial w_1}}{\frac{\partial y}{\partial C}} = -\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3w_1}\right)^{-\frac{2}{3}}\left(-\frac{1}{3w_1^2}\right)\left(\frac{2}{3w_2}\right)^{\frac{2}{3}} C}{\left(\frac{1}{3w_1}\right)^{\frac{1}{3}}\left(\frac{2}{3w_2}\right)^{\frac{2}{3}}} = \frac{C}{3w_1}$$

$$z_2(w,C) = -\frac{\frac{\partial y}{\partial w_2}}{\frac{\partial y}{\partial C}} = -\frac{\left(\frac{2}{3}\right)\left(\frac{1}{3w_1}\right)^{\frac{1}{3}}\left(\frac{2}{3w_2}\right)^{-\frac{1}{3}}\left(-\frac{2}{3w_2^2}\right) C}{\left(\frac{1}{3w_1}\right)^{\frac{1}{3}}\left(\frac{2}{3w_2}\right)^{\frac{2}{3}}} = \frac{2C}{3w_2}.$$

8. Consider a profit maximizing firm whose single-output technology is given by the production function $f:\mathbb{R}_+^L \to \mathbb{R}_+$ satisfying the usual assumptions (continuity, quasiconcavity, and monotonicity). Let $w \gg 0$ and $p > 0$ denote the input and the output prices, respectively. In the following, assume that the solution to the profit maximization problem is unique.

   (a) Show that the firm's profit function, $\pi(w,p)$, is a convex function of (w,p).
   Solution: Let $y(w,p)$ and $z(w,p)$ be the supply and the unconditional demand functions. Then,

$$\begin{aligned}\pi\left(\alpha(w,p)+(1-\alpha)(w',p')\right) &= \left(\alpha p+(1-\alpha)p'\right)y(w,p)-\left(\alpha w+(1-\alpha)w'\right)\cdot z(w,p)\\ &= \alpha\left(py(w,p)-w\cdot z(w,p)\right)+(1-\alpha)\left(p'y(w,p)-w'\cdot z(w,p)\right)\\ &\le \alpha\pi(w,p)+(1-\alpha)\pi(w',p'),\ \text{as required.}\end{aligned}$$

   (b) Show whether the firm's supply function $y(w,p)$ and the unconditional demand function $z(w,p)$ satisfy the law of supply and the law of demand, respectively. For this part, assume that $\pi(w,p)$, $y(w,p)$ and $z(w,p)$ are differentiable.
   Solution: By part (a), $\pi(w,p)$ is convex, which means $D_{(w,p)}\pi(w,p)$ is positive semidefinite. This, together with Hotelling's lemma yields,

$$\frac{\partial y(w,p)}{\partial p}=\frac{\partial^2\pi(w,p)}{\partial p^2}\ge 0 \quad\text{and}\quad \frac{\partial z_\ell(w,p)}{\partial w_\ell}=-\frac{\partial^2\pi(w,p)}{\partial w_\ell^2}\le 0,$$

   which are laws of supply and demand, respectively.

   (c) Show again whether the firm's supply function and the unconditional demand function satisfy the law of supply and the law of demand, respectively. For this part, do NOT assume that $\pi(w,p)$, $y(w,p)$ and $z(w,p)$ are differentiable.
   solution: Let $y=y(w,p)$, $z=z(w,p)$, $y'=y(w',p')$, and $z'=z(w',p')$. Then by the definition of profit maximization,

$$\begin{aligned}&(p,w)\cdot\left((y,-z)-(y'-z')\right)\ +\ (p',w')\cdot\left((y',-z')-(y-z)\right)\ \ge\ 0\\ \Longrightarrow\ &\left((p,w)-(p',w')\right)\cdot\left((y,-z)-(y'-z')\right)\ \ge\ 0.\end{aligned}$$

   The first component of the above (vector) inequality and the $\ell+1$-th component yields the law of supply and demand, respectively:

$$\begin{aligned}&\left(p-p'\right)\left(y-y'\right)\ge 0\\ &\left(w_\ell-w'_\ell\right)\left(-z_\ell-(-z'_\ell)\right)\ge 0 \Longrightarrow \left(w_\ell-w'_\ell\right)\left(z_\ell-z'_\ell\right)\le 0.\end{aligned}$$

   (d) Suppose there are J firms, and each firm's supply function satisfies the law of supply and the input demand function satisfies the law of demand. Show whether the aggregate supply and the aggregate demand satisfy the law of supply and demand, respectively.
   Solution: Suppose $p > p'$. Then $y_j(w,p)\ge y_j(w,p')$ for all j by the law of supply. Thus, $\sum_j y_j(w,p)\ge\sum_j y_j(w,p')$. Suppose $w_\ell > w'_\ell$ and $w_k = w'_k$ for all $k\ne\ell$. Then $z_{\ell j}(w,p)\le z_{\ell j}(w',p)$ for all j by the law of demand. Thus, $\sum_j z_{\ell j}(w,p)\le z_{\ell j}(w',p)$. Thus, the aggregate supply and the aggregate demand satisfies the laws of supply and demand, respectively.