Advanced Microeconomics I

Fall 2024 - M. Pak

Exercises: General equilibrium

1. Consider an Edgeworth box economy where preferences are given by

$$u_1(x_{11}, x_{21}) = 2\sqrt{x_{11}} + x_{21} \quad \text{and} \quad u_2(x_{12}, x_{22}) = 2\sqrt{x_{12}} + x_{22},$$

   (a) Suppose the initial endowments are $\omega_1 = (4,4)$ and $\omega_2 = (1,1)$. Find all the Pareto optimal allocations.

   Solution: Interior Pareto optimality is characterized by marginal rates of substitution being equal to each other. Thus,

$$MRS_1 = \frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{1}{\sqrt{x_{11}}} = \frac{1}{\sqrt{x_{12}}} = \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}} = MRS_2.$$

   Thus, $x_{11} = x_{12}$. Since $x_{11} + x_{12} = \bar{e}_1 = 5$, interior PO allocations are characterized by $x_{11} = x_{12} = 2.5$. So

$$\text{Interior PO set } = \{((2.5, x_{21}), (2.5, 5 - x_{21})) : 0 \leq x_{21} \leq 5\}.$$

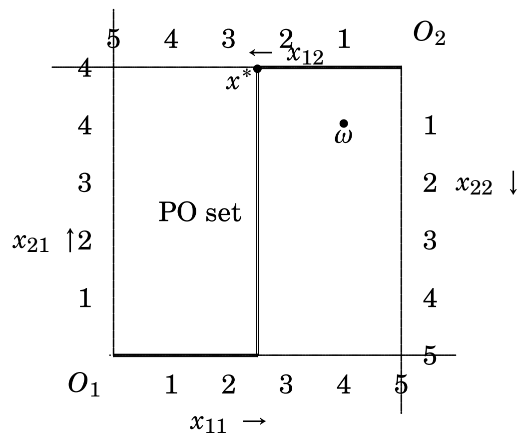
   Note that when $x_{11} < 2.5$ and $x_{21} = 0$ (which means $x_{12} > 2.5$ and $x_{22} = 5$), we have

$$MRS_1 = \frac{1}{\sqrt{x_{11}}} > \frac{1}{\sqrt{x_{12}}} = MRS_2.$$

   So this is a boundary PO allocation. Similarly, when $x_{11} > 2.5$ and $x_{21} = 5$ (which means $x_{12} < 2.5$ and $x_{22} = 0$), we have

$$MRS_1 = \frac{1}{\sqrt{x_{11}}} < \frac{1}{\sqrt{x_{12}}} = MRS_2.$$

   So this is a boundary PO allocations. The set of all the PO allocations are graphed below:



(b) Find the Walrasian equilibrium.

Solution: To find the Walrasian equilibrium, we first find the demand functions. Using normalization $p_2 = 1$,

$$MRS_1 = \frac{1}{\sqrt{x_{11}}} = \frac{p_1}{1} \implies x_{11} = \frac{1}{p_1^2}$$
$$MRS_2 = \frac{1}{\sqrt{x_{12}}} = \frac{p_1}{1} \implies x_{12} = \frac{1}{p_1^2}.$$

Market clearing condition for Market 1 yields

$$x_{11} + x_{12} = \bar{\omega}_1 \implies \frac{1}{p_1^2} + \frac{1}{p_1^2} = 5 \implies p_1^* = \sqrt{\frac{2}{5}} = 0.6325$$
$$x_{11}^* = \frac{1}{\frac{2}{5}} = 2.5 \quad \text{and} \quad x_{12}^* = \frac{1}{\frac{2}{5}} = 2.5$$

To find equilibrium amount of good 2, we use

$$\begin{aligned} p_1x_{11} + p_2x_{21} = 4p_1 + 4p_2 &\implies 0.6325(2.5) + x_{21} = 4(0.6325) + 4 \\ &\implies x_{21}^* = 4 + 1.5(0.6325) = 4 + 0.9488 = 4.9488 \\ p_1x_{12} + p_2x_{22} = 1p_1 + 1p_2 &\implies 0.6325(2.5) + x_{22} = 1(0.6325) + 1 \\ &\implies x_{22}^* = 1 - 1.5(0.6325) = 1 - 0.9488 = 0.0512. \end{aligned}$$

To recap, the equilibrium prices are $p^* = (0.6325, 1)$ and the equilibrium allocation is $\left(x^{1^*}, x^{2^*}\right) = \big((2.5, 4.9488), (2.5, 0.0512)\big)$.

2. Consider an Edgeworth box economy where preferences are given by

$$u_1(x_{11}, x_{21}) = \min\{x_{11}, x_{21}\} \quad \text{and} \quad u_2(x_{12}, x_{22}) = 2x_{12} + x_{22},$$

and the initial endowments are

$$\omega_1 = (3,2) \quad \text{and} \quad \omega_2 = (1,2).$$

Using the normalization $p_2 = 1$, find all the Walrasian equilibrium.

Solution: To find consumer 1's Marshallian demand function, we note that the two goods are perfect complements and that an optimal bundle must have $x_{11} = x_{21}$. So, substituting this into her budget equation yields:

$$p_1 x_{11} + p_2 x_{11} = 3p_1 + 2p_2 \implies x_1(p_1,p_2) = \left(\frac{3p_1+2p_2}{p_1+p_2}, \frac{3p_1+2p_2}{p_1+p_2}\right).$$

To find consumer 2's Marshallian demand function, we note that the two goods are perfect substitutes with the marginal rate of substitution equal to 2. So,

$$x_2(p_1,p_2) = \begin{cases} \left(\frac{p_1+2p_2}{p_1}, 0\right) & \text{if } \frac{p_1}{p_2} < 2, \\ \text{any } x_{12},\, x_{22}, \text{ s.t. } p_1 x_{12} + p_2 x_{22} = p_1 + 2p_2 & \text{if } \frac{p_1}{p_2} = 2 \\ \left(0, \frac{p_1+2p_2}{p_2}\right) & \text{if } \frac{p_1}{p_2} > 2. \end{cases}$$

Now, to find the market clearing prices we note the following. Consumer 1 never wants to consumer at the boundary. However, consumer 2 always wants to consume at the boundary unless $\frac{p_1}{p_2} = 2$. Therefore, the only possible candidate for an equilibrium price is $\frac{p_1}{p_2} = 2$.

To see, if $\frac{p_1}{p_2} = 2$ works, we note that at this price consumer 2 will be happy consuming anywhere on her budget line. So, we can clear the market by giving consumer 1 her desired consumption bundle and giving the remainder to consumer 2. That is, Walarsian equilibrium price is $p = (2,1)$, where we have normalized $p_2 = 1$ and the equilibrium allocation is given by

$$x_1 = x_1(2,1) = \left(\frac{8}{3}, \frac{8}{3}\right) \quad \text{and} \quad x_2 = \omega_1 + \omega_2 - x_1 = \left(\frac{4}{3}, \frac{4}{3}\right).$$

3. Suppose excess demand function $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$ satisfies Walras' Law (that is, $p \cdot z(p) = 0$ for all $p \in \Delta^\circ$). Show that if $z_\ell(p) = 0$ for any $L-1$ components of $z(p)$, then $z_\ell(p) = 0$ for all ℓ.

   Solution: By Walras' Law, we have

   $$p \cdot z(p) = p_1 z_1(p) + \cdots + p_{L-1} z_{L-1}(p) + p_L z_L(p) = 0.$$

   Suppose the markets for $L-1$ goods, say goods $\ell = 1, \ldots, L-1$ clears. Then $p_L z_L(p) = 0$. Since $p_L > 0$, we have $z_L(p) = 0$.

4. Consider again the economy given in Problem set 5, question 2.

   (a) Find all the Walrasian equilibrium using the normalization $p_2 = 1$.
   Solution: We solve for the market clearing condition for good 2:

$$x_2(w,1) = \omega_2 + y(w,1) \quad \Longrightarrow \quad \frac{5w + \frac{1}{4w}}{3} = \frac{1}{2w} \quad \Longrightarrow \quad w^* = \frac{1}{2}.$$

   Thus, Walrasian equilibrium price is $(w^*, p_2^*) = (\frac{1}{2}, 1)$, and the corresponding Walrasian equilibrium consumption and production plans are $(x_1^*, x_2^*) = (4,1)$ and $(z^*, y^*) = (1,1)$.

   (b) Using a single diagram, graph the consumer's utility maximizing-consumption plan and the firm's profit-maximizing production plan for the equilibrium found above.
   Solution:

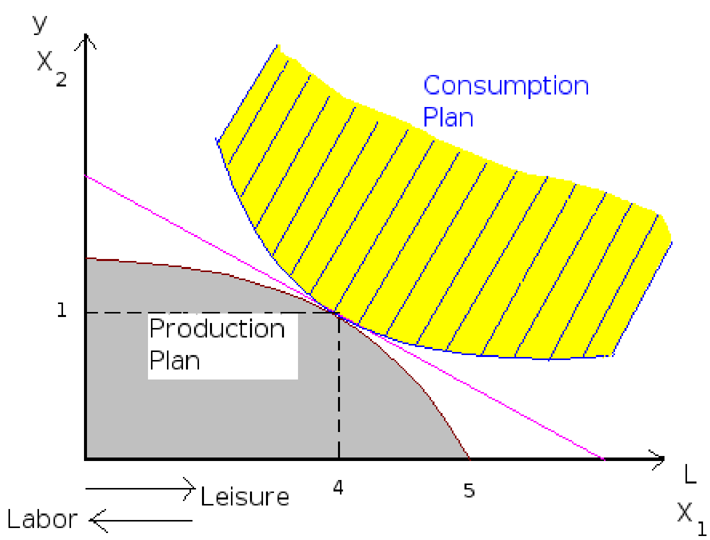


Figure 1: Production and Consumption Plan at equilibrium price.

5. Show that the firms' profit maximizing production plans are homogeneous of degree zero in an Arrow-Debreu economy, while the profit functions are homogeneous of degree one. That is, for all $\alpha > 0$, $y_j(\alpha p) = y_j(p)$ and $\pi_j(\alpha p) = \alpha \pi_j(p)$. Also show that the individuals' demand correspondences are homogeneous of degree zero.

   Solution: For all $\alpha > 0$, we have

$$\max_{y \in Y_J} (\alpha p) \cdot y_j \iff \max_{y \in Y_J} \alpha (p \cdot y_j) \iff \max_{y \in Y_J} p \cdot y_j.$$

   Therefore, $y_j(\alpha p) = y_j(p)$. This then implies that

$$\pi_j(\alpha p) = (\alpha p) \cdot y_j(\alpha p) = (\alpha p) \cdot y_j(p) = \alpha (p \cdot y_j(p)) = \alpha \pi_j(p).$$

Individual i's budget set when the price vector is αp is

$$
\begin{aligned}
(\alpha p)\cdot x_i \le (\alpha p)\cdot \omega_i + \pi(\alpha p) &\Longleftrightarrow (\alpha p)\cdot x_i \le (\alpha p)\cdot \omega_i + \alpha\pi(p)\\
&\Longleftrightarrow p\cdot x_i \le p\cdot \omega_i + \pi(p),
\end{aligned}
$$

which is the same budget set as when the price is p. Therefore, $x_i(\alpha p) = x_i(p)$.

6. Consider a 2-goods economy with 2 consumers and 1 firm. The consumers have identical preferences:

$$u_i(x_{1i}, x_{2i}) = x_{1i}^{\frac{1}{2}} x_{2i}^{\frac{1}{2}},$$

while their endowments are $\omega_1 = (10,0)$ and $\omega_2 = (0,10)$. The firm uses the first good as input to produce the second good. Its production function is

$$f(z) = z^{\frac{1}{2}},$$

where z denotes the amount of good 1. Let $\theta_1 = \theta_2 = \frac{1}{2}$ be the consumers' shares of the firm.

(a) Find the Walrasian equilibrium (use the normalization $p_2 = 1$).

Solution: Letting $p_2 = 1$, the firm's profit maximization problem is

$$\max_z \; z^{\frac{1}{2}} - p_1 z.$$

The first order condition yields

$$
\begin{aligned}
\tfrac{1}{2}z^{-\frac{1}{2}} - p_1 \;=\; 0 \Longrightarrow z^{-\frac{1}{2}} \;=\; 2p_1 \Longrightarrow z(p) \;&=\; \frac{1}{4p_1^2}\\
\Longrightarrow y(p) \;=\; z(p)^{\frac{1}{2}} \;&=\; \frac{1}{2p_1}\\
\Longrightarrow \pi(p) \;=\; z(p)^{\frac{1}{2}} - p_1 z(p_1) \;&=\; \frac{1}{2p_1} - \frac{1}{4p_1} \;=\; \frac{1}{4p_1}.
\end{aligned}
$$

Next, consumer i's income is

$$I_1 = p\cdot\omega_1 + \frac{\pi(p)}{2} = 10p_1 + \frac{1}{8p_1} \quad \text{and} \quad I_2 = p\cdot\omega_2 + \frac{\pi(p)}{2} = 10 + \frac{1}{8p_1}$$

Using the well-known formula for Cobb-Douglas utility function, we obtain

$$
\begin{aligned}
x_{11}(p) &= \frac{I_1}{2p_1} = \frac{10p_1 + \frac{1}{8p_1}}{2p_1} = 5 + \frac{1}{16p_1^2}\\
x_{21}(p) &= \frac{I_1}{2} = \frac{10p_1 + \frac{1}{8p_1}}{2} = 5p_1 + \frac{1}{16p_1}\\
x_{12}(p) &= \frac{I_2}{2p_1} = \frac{10 + \frac{1}{8p_1}}{2p_1} = \frac{5}{p_1} + \frac{1}{16p_1^2}\\
x_{22}(p) &= \frac{I_2}{2} = \frac{10 + \frac{1}{8p_1}}{2} = 5 + \frac{1}{16p_1}.
\end{aligned}
$$

Thus, the market clearing condition for good 2 is

$$\begin{aligned}
x_{11}(p)+x_{12}(p) &= \bar{\omega}_2 + y(p) \\
5p_1 + \frac{1}{16p_1} + 5 + \frac{1}{16p_1} &= 10 + \frac{1}{2p_1} \\
80p_1^2 + 1 + 80p_1 + 1 &= 160p_1 + 8 \\
80p_1^2 - 80p_1 - 6 &= 0 \\
40p_1^2 - 40p_1 - 3 &= 0
\end{aligned}$$

$$\begin{aligned}
p_1 &= \frac{40 \pm \sqrt{40^2 - 4(40)(-3)}}{2(40)} = \frac{40 \pm \sqrt{1600+480}}{80} \\
&= \frac{40 \pm \sqrt{2080}}{80} = \frac{40 \pm 45.61}{80} = 1.07.
\end{aligned}$$

Substituting $p_1 * = 1.07$ in the demand functions yields

$$\begin{aligned}
x_{11}^* &= 5 + \frac{1}{16(1.07)^2} = 5.05 \quad \text{and} \quad x_{21}^* = 5p_1 + \frac{1}{16(1.07)} = 5.41 \\
x_{12}^* &= \frac{5}{p_1} + \frac{1}{16(1.07)^2} = 4.73 \quad \text{and} \quad x_{22}^* = 5 + \frac{1}{16(1.07)} = 5.06 \\
z^* &= \frac{1}{4(1.07)^2} = 0.22 \quad \text{and} \quad y^* = \frac{1}{2(1.07)} = 0.47.
\end{aligned}$$

(b) Verify that the first fundamental of theorem of welfare holds.

Solution: At the equilibrium,

$$\begin{aligned}
MRS_1 &= \frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{x_{21}^*}{x_{11}^*} = \frac{5.41}{5.05} = 1.07 \\
MRS_2 &= \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}} = \frac{x_{22}^*}{x_{12}^*} = \frac{5.06}{4.73} = 1.07.
\end{aligned}$$

Thus, the indifference curves of the two consumers are tangent to each other, and we have Pareto optimality.

(c) Which consumer, if any, will be worse off if there was no firm (that is, if this was a pure exchange economy)? Give an interpretation of this result.

Solution: Let the double star ($**$) represent the equilibrium without the firm. Solving the market clearing condition for the second market now yields

$$5p_1^{**} + 5 = 10 \implies p_1^{**} = 1 \implies x_{\ell i}^{**} = 5 \text{ for all } \ell \text{ and } i.$$

Therefore,

$$\begin{aligned}
u_1(x_{11}^{**}, x_{21}^{**}) &= \sqrt{5(5)} = 5 < 5.23 = \sqrt{5.05(5.41)} = u_1(x_{11}^*, x_{21}^*) \\
u_2(x_{12}^{**}, x_{22}^{**}) &= \sqrt{5(5)} = 5 > 4.89\sqrt{4.73(5.06)} = u_1(x_{11}^*, x_{21}^*).
\end{aligned}$$

Note that possibility of production makes the input good (good 1) more valuable (more scarce) and makes the output good (good 2) less valuable (more abundant). Since consumer 1 is the only supplier of good 1 while consumer 2 is the only supplier of good 2, it makes sense that consumer is better off with the production while consumer 2 is not.

7. Consider a 2-goods economy with 2 consumers and 1 firm. The consumers have identical preferences and endowments:

$$u_i(x_{1i}, x_{2i}) = x_{1i} + \ln x_{2i} \quad \text{and} \quad \omega_i = (10, 0), \qquad i = 1, 2.$$

The firm uses the first good as input to produce the second good. Its production function is

$$f(v) = v^{\frac{1}{2}},$$

where v denotes the amount of good 1 the firm uses as input (we are using v rather than the typical z to denote the input good since $z(p)$ is used for the market excess demand function). Let θ_i denote consumer i's share of the firm.

(a) Normalizing $p_2 = 1$, find the market excess demand function.

Solution: Letting $p_2 = 1$, the firm's profit maximization problem is

$$\max_v \; v^{\frac{1}{2}} - p_1 v.$$

The first order condition yields

$$\begin{aligned}
\tfrac{1}{2}v^{-\frac{1}{2}} - p_1 &= 0 \Longrightarrow v^{-\frac{1}{2}} = 2p_1 \Longrightarrow v(p) = \frac{1}{4p_1^2} \\
\Longrightarrow y(p) &= v(p_1)^{\frac{1}{2}} = \frac{1}{2p_1} \\
\Longrightarrow \pi(p) &= v(p_1)^{\frac{1}{2}} - p_1 v(p_1) = \frac{1}{2p_1} - \frac{1}{4p_1} = \frac{1}{4p_1}.
\end{aligned}$$

Next, consumer i's utility maximization problem is

$$\max_{x_{1i}, x_{2i}} \; x_{1i} + \ln x_{2i} \quad \text{s.t.} \quad p_1 x_{1i} + x_{2i} = 10p_1 + \frac{\theta_i}{4p_1}.$$

The solution must satisfy:

$$\begin{aligned}
\frac{\frac{\partial u_i}{\partial x_{1i}}}{\frac{\partial u_i}{\partial x_{2i}}} &= \frac{1}{\frac{1}{x_{2i}}} = \frac{p_1}{1} \\
\Longrightarrow x_{2i}(p) &= p_1 \\
\Longrightarrow x_{1i}(p) &= \frac{10p_1 + \frac{\theta_i}{4p_1} - p_1}{p_1} = 9 + \frac{\theta_i}{4p_1^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
z_1(p) &= x_{11}(p)+x_{12}(p)+v(p)-\bar{\omega}_1 \;=\; 18+\frac{\theta_1+\theta_2}{4p_1^2}+\frac{1}{4p_1^2}-20 \\
&= \frac{1}{2p_1^2}-2 \\
z_2(p) &= x_{21}(p)+x_{22}(p)-y(p)-\bar{\omega}_2 \;=\; 2p_1-\frac{1}{2p_1}.
\end{aligned}$$

(b) Find the Walrasian equilibrium price, allocations, and production plans. How do the equilibrium price and allocations depend on θ_1? Given an intuitive explanation for these results in words.

Solution: Market clearing condition for the second good yields:

$$z_2(p)=0 \iff 2p_1 = \frac{1}{2p_1} \iff p_1^2 = \frac{1}{4} \implies p_1^* = \frac{1}{2}.$$

Thus, the equilibrium price is $p^* = (\frac{1}{2},1)$ and the equilibrium allocations are:

$$\begin{aligned}
x_1^* &= \left(9+\theta_1,\ \frac{1}{2}\right) \\
x_2^* &= \left(9+(1-\theta_1),\ \frac{1}{2}\right) \\
(v^*,y^*) &= (1,\ 1).
\end{aligned}$$

As seen above, θ_1 does not affect the equilibrium price or the equilibrium allocation for the second good. This is because the preferences are quasi-linear with respect to the first good, which means that any wealth effects are absorbed by the demand for the first good. That is, changes in wealth do not affect the demand for good 2. Thus, since changes in θ_1 only affects the wealth of the consumers, its effects are absorbed by the first good. In particular, since increase in θ_1 increases consumer 1's wealth and decreases consumer 2's wealth, x_{11}^* is increasing in θ_1 and x_{12}^* is decreasing in θ_1.

Since this is a two goods economy, $\hat{z}(p)=z_1(p_1)$. We have

8. Consider a pure exchange economy with 2 consumers and two goods. Suppose initially, the consumers' endowment allocation is $\omega' = (\omega_1', \omega_2') = \big((8,0),(0,4)\big)$, and the equilibrium price is $p'=(1,1)$. However, there is a shock to the economy that causes the endowment allocation to change to $\omega'' = (\omega_1'', \omega_2'') = \big((2,6),(0,2)\big)$. Can you predict what will happen to the equilibrium price ratio? That is, will the equilibrium price ratio $\frac{p_1''}{p_2''}$ be larger, smaller, or same as $\frac{p_1'}{p_2'}$? Assume that the consumers' preferences are continuous, strongly monotone, and strictly convex and that the preferences are not affected by the shock.

Solution: Let x_1' be the consumer 1's bundle in the original equilibrium. As seen in the graph, x_1' must necessarily lie to the right of ω_1'', which means consumer 1's indifference curve going through her new endowment ω_i'' must be stepper than $\frac{p_1'}{p_2'} = 1$. So the new equilibrium cannot have a budget line flatter than the orginal budget line. That is the new budget line in equilibrium must be steeper, or $\frac{p_1''}{p_2''} > \frac{p_1'}{p_2'}$.

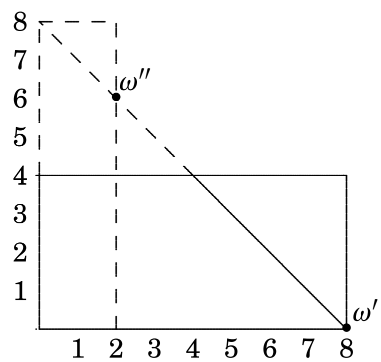


Figure 2: Question 4.