Advanced Microeconomics I

Fall 2024 - M. Pak

Exercises: Consumer Theory and Aggregate Demand

1 Consumer Theory

1. Show the following:

   (a) If $\succsim$ is strongly monotone, then it is monotone.
   Solution: Let $\succsim$ be strongly monotone. To show that $\succsim$ is monotone, we need to show that $x \gg y \Rightarrow x \succ y$. But, this is trivial since $x \gg y \Rightarrow x \geq y \Rightarrow x \succ y$ by strong monotonicity.

   (b) If $\succsim$ is monotone, then it is locally non-satiated.
   Solution: Let $\succsim$ be monotone. Take any bundle x and $\varepsilon > 0$. No matter how small ε, there is always $y \gg x$ such that $\|y - x\| < \varepsilon$. By weak monotonicity $y \succ x$. Therefore, $\succsim$ is locally non-satiated as well.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. That is, $f(a) > f(b)$ if and only if $a > b$.

   (a) Show that if $u(x)$ is a utility function representing a preference relation $\succsim$, then the function $\tilde{u}(x) = f(u(x))$ also represents $\succsim$.
   Solution: To show that $\tilde{u}(x)$ also represents $\succsim$, we need to show that $\tilde{u}(x) \geq \tilde{u}(y)$ if and only if $x \succsim y$. To see this, notice that

$x \succsim y$	if and only if	$u(x) \geq u(y)$	since $u(\cdot)$ represents $\succsim$.
	if and only if	$f(u(x)) \geq f(u(y))$	since $f(\cdot)$ is strictly increasing.
	if and only if	$\tilde{u}(x) \geq \tilde{u}(y)$	by the definition of $\tilde{u}(\cdot)$.

   (b) Let $x(p,w)$ be the Marshallian demand for utility function $u(x)$, and let $\tilde{x}(p,w)$ be the Marshallian demand for utility function $\tilde{u}(x)$, where $\tilde{u}(x)$ is as in part (a). Show that $x(p,w) = \tilde{x}(p,w)$.
   Solution: Let $B(p,w) = \{x : p \cdot x \leq w\}$ be the budget set. Since functions $u(x)$ and $\tilde{u}(x)$ represents the same preference $\succsim$, we have

$$\begin{aligned} x(p,w) &= \arg\max_x \; u(x) \text{ s.t. } p \cdot x \leq w \\ &= \{x^* \in B(p,w) \text{ s.t. } x^* \succsim x \text{ for all } x \in B(p,w)\} \\ &= \arg\max_x \; \tilde{u}(x) \text{ s.t. } p \cdot x \leq w \\ &= \tilde{x}(p,w) \end{aligned}$$

(c) Let $v(p,w)$ be an indirect utility function, and let $\tilde{v}(p,w) = f(v(p,w))$ be an increasing transformation of $v(\cdot)$. Explain whether the Marshallian demand corresponding to these two indirect utility functions are the same. You may assume as much differentiability as needed.

Solution: Let $\tilde{x}(p,w)$ and $x(p,w)$ be the Marshallian demand functions corresponding to $\tilde{v}(p,w)$ and $v(p,w)$, respectively. Applying Roy's Identity yields

$$\begin{aligned} \tilde{x}_\ell(p,w) &= -\frac{\frac{\partial f(v(p,w))}{\partial p_\ell}}{\frac{\partial f(v(p,w))}{\partial w}} = -\frac{\frac{\partial f(v(p,w))}{\partial v}\left(\frac{\partial v(p,w)}{\partial p_\ell}\right)}{\frac{\partial f(v(p,w))}{\partial v}\left(\frac{\partial v(p,w)}{\partial w}\right)} \\ &= -\frac{\frac{\partial v(p,w)}{\partial p_\ell}}{\frac{\partial v(p,w)}{\partial w}} = x_\ell(p,w). \end{aligned}$$

3. Let $\succsim$ be a continuous, homothetic preference. Note that a continuous $\succsim$ is homothetic if and only if it admits a utility function that is homogeneous of degree one (see MWG Exercise 3.C.5).

(a) Give an example of a utility function for $\succsim$ that is homogeneous of degree one and one that is not.

Solution: A Cobb-Douglas utility function in the standard form is HD1. For example, let $u(x_1,x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. Then

$$u(\alpha x_1, \alpha x_2) = (\alpha x_1)^{\frac{1}{2}}(\alpha x_2)^{\frac{1}{2}} = \alpha u(x_1,x_2).$$

But an increasing transformation of a standard Cobb-Douglas utility function may not be HD1. For example, $\tilde{u}(x_1,x_2) = (u(x_1,x_2))^2 = x_1 x_2$ is HD2:

$$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) = \alpha^2 u(x_1,x_2).$$

(b) Assuming that $u(\cdot)$ is a differentiable utility function representing a homothetic preference, show that the marginal rate of substitution (MRS) at x is the same as the MRS at αx for any $\alpha > 0$.

Solution: Applying HD1, we obtain

$$\frac{\frac{\partial u(\alpha x)}{\partial x_\ell}}{\frac{\partial u(\alpha x)}{\partial x_k}} = \frac{\frac{\partial \alpha u(x)}{\partial x_\ell}}{\frac{\partial \alpha u(x)}{\partial x_k}} = \frac{\frac{\partial u(x)}{\partial x_\ell}}{\frac{\partial u(x)}{\partial x_k}}.$$

(c) By *wealth expansion path* we mean the curve traced out by $x(p,w)$ as w varies. Show that the wealth expansion path of $u(\cdot)$ is a ray that starts from the origin.

Solution: To see this, note that Marshallian demand at wealth $w = 1$, $x^* = x(p,1)$, satisfies the first order conditions $p \cdot x^* = 1$, and

$$\text{for all } \ell,\ \frac{\partial u(x^*)}{\partial x_\ell} \le \lambda p_\ell, \text{ (with equality if } x_\ell^* > 0).$$

Let $u(\cdot)$ be HD1. Then since $\frac{\partial u(wx^*)}{\partial p_\ell} = \frac{w\partial u(x^*)}{\partial p_\ell}$, $\hat{x} = wx^*$ satisfies the first order conditions $p \cdot \hat{x} = w$, and

$$\text{for all } \ell, \ \frac{\partial u(\hat{x})}{\partial x_\ell} \leq \hat{\lambda} p_\ell, \text{ (with equality if } \hat{x}_\ell > 0).$$

Therefore, $x(p,w) = wx^*$.

4. Let $u(x) = \sqrt{x_1} + x_2$.

   (a) Find the Marshallian demand function. (Please pay attention to the possibility of boundary solutions).
   Solution: Let

$$\mathscr{L}(x_1, x_2, \lambda) \ = \ \sqrt{x_1} + x_2 + \lambda[w - p_1 x_1 - p_2 x_2].$$

The first order conditions are:

$$(1) \qquad \frac{\partial \mathscr{L}}{\partial x_1} = \frac{1}{2\sqrt{x_1^*}} - \lambda p_1 \leq 0 \quad (= 0 \text{ if } x_1^* > 0)$$

$$(2) \qquad \frac{\partial \mathscr{L}}{\partial x_2} = 1 - \lambda p_2 \leq 0 \quad (= 0 \text{ if } x_2^* > 0)$$

$$(3) \qquad \frac{\partial \mathscr{L}}{\partial \lambda} = w - p_1 x_1^* - p_2 x_2^* = 0.$$

Assuming interior solution and dividing (2) by (1) yields

$$2\sqrt{x_1^*} = \frac{p_2}{p_1} \ \Rightarrow \ x_1^* = \frac{p_2^2}{4p_1^2}.$$

Substitute into the budget equation to obtain

$$p_1\left(\frac{p_2^2}{4p_1^2}\right) + p_2 x_2^* = w \ \Rightarrow \ x_2^* = \frac{w - \frac{p_2^2}{4p_1}}{p_2} = \frac{w}{p_2} - \frac{p_2}{4p_1}. \quad (*)$$

From the expression $(*)$ above, we see that an interior solution is obtained only if $p_1 > \frac{p_2^2}{4w}$. When, $p_1 \leq \frac{p_2^2}{4w}$, $(*)$ implies $x_2^* \leq 0$, in which case the non-negativity constraint will force $x_2^* = 0$ and $x_1^* = \frac{w}{p_1}$. To summarize, the Marshallian demand is given by:

$$x(p,w) = \begin{cases} \left(\frac{p_2^2}{4p_1^2}\,,\ \frac{w}{p_2} - \frac{p_2}{4p_1}\right) & \text{if } p_1 > \frac{p_2^2}{4w}, \quad \text{and} \\ \\ \left(\frac{w}{p_1}\,,\ 0\right) & \text{if } p_1 \leq \frac{p_2^2}{4w}. \end{cases}$$

   (b) Verify that the Marshallian demand found above is homogeneous of degree zero in (p,w) and satisfies the Walras' Law.

Solution: We have

$$
\begin{aligned}
x(\alpha p, \alpha w) &= \begin{cases} \left(\frac{(\alpha p_2)^2}{4(\alpha p_1)^2}\ ,\ \frac{\alpha w}{\alpha p_2} - \frac{\alpha p_2}{4(\alpha p_1)}\right) & \text{if } \alpha p_1 > \frac{(\alpha p_2)^2}{4(\alpha w)}, \quad \text{and} \\ \left(\frac{\alpha w}{\alpha p_1}\ ,\ 0\right) & \text{if } \alpha p_1 \le \frac{(\alpha p_2)^2}{4(\alpha w)} \end{cases} \\
&= \begin{cases} \left(\frac{p_2^2}{4p_1^2}\ ,\ \frac{w}{p_2} - \frac{p_2}{4p_1}\right) & \text{if } p_1 > \frac{p_2^2}{4w}, \quad \text{and} \\ \left(\frac{w}{p_1}\ ,\ 0\right) & \text{if } p_1 \le \frac{p_2^2}{4w} \end{cases} \\
&= x(p,w),
\end{aligned}
$$

and

$$
\begin{aligned}
p \cdot x(p,w) &= \begin{cases} \frac{p_1 p_2^2}{4p_1^2} + \frac{p_2 w}{p_2} - \frac{p_2^2}{4p_1} & \text{if } p_1 > \frac{p_2^2}{4w}, \quad \text{and} \\ \frac{p_1 w}{p_1} + p_2(0) & \text{if } p_1 \le \frac{p_2^2}{4w}. \end{cases} \\
&= w.
\end{aligned}
$$

(c) Find the indirect utility function.
Solution:

$$
\begin{aligned}
v(p,w) = \sqrt{x_1(p,w)} + x_2 &= \begin{cases} \sqrt{\frac{p_2^2}{4p_1^2}} + \frac{w}{p_2} - \frac{p_2}{4p_1} & \text{if } p_1 > \frac{p_2^2}{4w}, \quad \text{and} \\ \sqrt{\frac{w}{p_1}} & \text{if } p_1 \le \frac{p_2^2}{4w}. \end{cases} \\
&= \begin{cases} \frac{p_2}{4p_1} + \frac{w}{p_2} & \text{if } p_1 > \frac{p_2^2}{4w}, \quad \text{and} \\ \sqrt{\frac{w}{p_1}} & \text{if } p_1 \le \frac{p_2^2}{4w}. \end{cases}
\end{aligned}
$$

(d) Verify that the indirect utility function is homogenous of degree zero in (p,w), strictly increasing in w and non-increasing in p_ℓ for all ℓ.
Solution: Verifying that $v(p,w)$ HD0, strictly increasing in w and non-increasing in p_1 is trivial. To check that it is non-increasing in p_2, we differentiate $v(p,w)$ w.r.t. p_2:

$$
\frac{\partial v(p,w)}{\partial p_2} = \begin{cases} \frac{1}{4p_1} - \frac{w}{p_2^2} \text{ (which is negative)} & \text{if } p_1 > \frac{p_2^2}{4w}, \quad \text{and} \\ 0 & \text{if } p_1 \le \frac{p_2^2}{4w}. \end{cases}
$$

5. Let $u(x) = x_1 + \ln x_2$.

(a) Find the Marshallian demand function. (Please pay attention to the possibility of boundary solutions).

Solution: Let

$$\mathscr{L}(x_1,x_2,\lambda) \;=\; x_1 + \ln x_2 + \lambda[w - p_1x_1 - p_2x_2].$$

The first order conditions are:

$$\begin{aligned}
&(1) \qquad \frac{\partial \mathscr{L}}{\partial x_1} = 1 - \lambda p_1 \le 0 \quad (= 0 \text{ if } x_1^* > 0)\\
&(2) \qquad \frac{\partial \mathscr{L}}{\partial x_2} = \frac{1}{x_2^*} - \lambda p_2 \le 0 \quad (= 0 \text{ if } x_2^* > 0)\\
&(3) \qquad \frac{\partial \mathscr{L}}{\partial \lambda} = w - p_1x_1^* - p_2x_2^* = 0.
\end{aligned}$$

Assuming interior solution and dividing (1) by (2) yields

$$x_2^* = \frac{p_1}{p_2}$$

Substitute into the budget equation to obtain

$$(*) \quad x_1^* = \frac{w}{p_1} - 1.$$

From the expression $(*)$ above, we see that an interior solution is obtained only if $p_1 < w$. When, $p_1 \ge w$, $(*)$ implies $x_1^* \le 0$, in which case the non-negativity constraint will force $x_1^* = 0$ and $x_2^* = \frac{w}{p_2}$. To summarize, the Marshallian demand is given by:

$$x(p,w) = \begin{cases} \left(\frac{w}{p_1} - 1\,,\; \frac{p_1}{p_2}\right) & \text{if } p_1 < w, \quad \text{and} \\[2ex] \left(0\,,\; \frac{w}{p_2}\right) & \text{if } p_1 \ge w. \end{cases}$$

(b) Verify that the Marshallian demand found above is homogeneous of degree zero in (p,w) and satisfies the Walras' Law.

Solution: To verify HD0, let $\alpha > 0$. Then

$$x(\alpha p, \alpha w) = \begin{cases} \left(\frac{\alpha w}{\alpha p_1} - 1\,,\; \frac{\alpha p_1}{\alpha p_2}\right) = \left(\frac{w}{p_1} - 1\,,\; \frac{p_1}{p_2}\right) & \text{if } p_1 < w, \quad \text{and} \\[2ex] \left(\alpha(0)\,,\; \frac{\alpha w}{\alpha p_2}\right) = \left(0\,,\; \frac{w}{p_2}\right) & \text{if } p_1 \ge w. \end{cases}$$

To verify Walras' Law

$$p \cdot x(p,w) = \begin{cases} (p_1,p_2)\cdot\left(\frac{w}{p_1} - 1\,,\; \frac{p_1}{p_2}\right) = w & \text{if } p_1 < w, \quad \text{and} \\[2ex] (p_1,p_2)\cdot\left(0\,,\; \frac{w}{p_2}\right) = w & \text{if } p_1 \ge w. \end{cases}$$

(c) Find the indirect utility function.

Solution: Since $v(p,w) = u(x(p,w))$, we have

$$v(p,w) = \begin{cases} \left(\frac{w}{p_1} - 1\right) + \ln\left(\frac{p_1}{p_2}\right) & \text{if } p_1 < w, \quad \text{and} \\[2ex] \ln\left(\frac{w}{p_2}\right) & \text{if } p_1 \ge w. \end{cases}$$

(d) Verify that the indirect utility function is homogenous of degree zero in (p,w), strictly increasing in w and non-increasing in p_ℓ for all ℓ.

Solution: To verify HD0, let $\alpha > 0$. Then

$$\begin{aligned} v(\alpha p, \alpha w) &= \begin{cases} \left(\frac{\alpha w}{\alpha p_1} - 1\right) + \ln\left(\frac{\alpha p_1}{\alpha p_2}\right) & \text{if } \alpha p_1 < \alpha w, \quad \text{and} \\ \ln\left(\frac{\alpha w}{\alpha p_2}\right) & \text{if } \alpha p_1 \geq \alpha w. \end{cases} \\ &= \begin{cases} \left(\frac{w}{p_1} - 1\right) + \ln\left(\frac{p_1}{p_2}\right) & \text{if } p_1 < w, \quad \text{and} \\ \ln\left(\frac{w}{p_2}\right) & \text{if } p_1 \geq w. \end{cases} \\ &= v(p,w) \end{aligned}$$

To verify that $v(p,w)$ is strictly increasing in w:

$$\frac{\partial v(p,w)}{\partial w} = \begin{cases} \frac{1}{p_1} > 0 & \text{if } p_1 < w, \quad \text{and} \\ \left(\frac{p_2}{w}\right)\left(\frac{1}{p_2}\right) = \frac{1}{w} > 0 & \text{if } p_1 \geq w. \end{cases}$$

So, $v(p,w)$ is strictly increasing in w.

Next,

$$\frac{\partial v(p,w)}{\partial p_2} = \begin{cases} -\frac{p_1}{p_2^2} < 0 & \text{if } p_1 < w, \quad \text{and} \\ -\frac{w}{p_2^2} < 0 & \text{if } p_1 \geq w. \end{cases}$$

So, $v(p,w)$ is non-increasing (in fact, strictly decreasing) in p_2. To check that $v(p,w)$ is non-increasing in p_1, we differentiate $v(p,w)$ at $p_1 \neq w$. Then,

$$\frac{\partial v(p,w)}{\partial p_1} = \begin{cases} -\frac{w}{p_1^2} + \frac{1}{p_1} = \left(1 - \frac{w}{p_1}\right)\frac{1}{p_1} < 0 & \text{if } p_1 < w, \quad \text{and} \\ 0 & \text{if } p_1 > w. \end{cases}$$

So, $v(p,w)$ is non-increasing everywhere, except may be at $p_1 = w$. But, since $v(p,w)$ is continuous, it is actually non-increasing everywhere.

6. Let $u(x) = x_1 + 2\ln x_2$.

(a) Find the Marshallian demand function. (Please pay attention to the possibility of boundary solutions).

Solution: Let

$$\mathscr{L}(x_1, x_2, \lambda) = x_1 + 2\ln x_2 + \lambda[w - p_1 x_1 - p_2 x_2].$$

The first order conditions are:

$$\begin{aligned}
&(1) \qquad \frac{\partial \mathscr{L}}{\partial x_1} = 1 - \lambda p_1 \le 0 \quad (= 0 \text{ if } x_1^* > 0) \\
&(2) \qquad \frac{\partial \mathscr{L}}{\partial x_2} = \frac{2}{x_2^*} - \lambda p_2 \le 0 \quad (= 0 \text{ if } x_2^* > 0) \\
&(3) \qquad \frac{\partial \mathscr{L}}{\partial \lambda} = w - p_1 x_1^* - p_2 x_2^* = 0.
\end{aligned}$$

Assuming interior solution and dividing (1) by (2) yields

$$x_2^* = \frac{2p_1}{p_2}$$

Substitute into the budget equation to obtain

$$(*) \quad x_1^* = \frac{w - 2p_1}{p_1}.$$

From the expression $(*)$ above, we see that an interior solution is obtained only if $p_1 < \frac{w}{2}$. When $p_1 \ge \frac{w}{2}$, $(*)$ implies $x_1^* \le 0$, in which case the non-negativity constraint will force $x_1^* = 0$ and $x_2^* = \frac{w}{p_2}$. To summarize, the Marshallian demand is given by:

$$x(p,w) = \begin{cases} \left(\frac{w-2p_1}{p_1}, \frac{2p_1}{p_2}\right) & \text{if } p_1 < \frac{w}{2}, \quad \text{and} \\[2ex] \left(0, \frac{w}{p_2}\right) & \text{if } p_1 \ge \frac{w}{2}. \end{cases}$$

(b) Verify that the Marshallian demand found above is homogeneous of degree zero in (p,w) and satisfies the Walras' Law.

Solution: To verify HD0, let $\alpha > 0$. Then

$$x(\alpha p, \alpha w) = \begin{cases} \left(\frac{\alpha w-2\alpha p_1}{\alpha p_1}, \frac{2\alpha p_1}{\alpha p_2}\right) = \left(\frac{w-2p_1}{p_1}, \frac{2p_1}{p_2}\right) & \text{if } p_1 < \frac{w}{2}, \quad \text{and} \\[2ex] \left(\alpha(0), \frac{\alpha w}{\alpha p_2}\right) = \left(0, \frac{w}{p_2}\right) & \text{if } p_1 \ge \frac{w}{2}. \end{cases}$$

To verify Walras' Law

$$p \cdot x(p,w) = \begin{cases} (p_1,p_2)\cdot\left(\frac{w-2p_1}{p_1}, \frac{2p_1}{p_2}\right) = w & \text{if } p_1 < \frac{w}{2}, \quad \text{and} \\[2ex] (p_1,p_2)\cdot\left(0, \frac{w}{p_2}\right) = w & \text{if } p_1 \ge \frac{w}{2}. \end{cases}$$

(c) Find the indirect utility function.

Solution: Since $v(p,w) = u(x(p,w))$, we have

$$v(p,w) = \begin{cases} \frac{w-2p_1}{p_1} + 2\ln\left(\frac{2p_1}{p_2}\right) & \text{if } p_1 < \frac{w}{2}, \quad \text{and} \\[2ex] 2\ln\left(\frac{w}{p_2}\right) & \text{if } p_1 \ge \frac{w}{2}. \end{cases}$$

(d) Verify that the indirect utility function is homogenous of degree zero in (p,w), strictly increasing in w and non-increasing in p_ℓ for all ℓ.

Solution: To verify HD0, let $\alpha > 0$. Then

$$\begin{aligned} v(\alpha p, \alpha w) &= \begin{cases} \frac{\alpha w - 2\alpha p_1}{\alpha p_1} + 2\ln\left(\frac{2\alpha p_1}{\alpha p_2}\right) & \text{if } \alpha p_1 < \frac{\alpha w}{2}, \quad \text{and} \\ 2\ln\left(\frac{\alpha w}{\alpha p_2}\right) & \text{if } \alpha p_1 \geq \frac{\alpha w}{2}. \end{cases} \\ &= \begin{cases} \frac{w - 2p_1}{p_1} + 2\ln\left(\frac{2p_1}{p_2}\right) & \text{if } p_1 < \frac{w}{2}, \quad \text{and} \\ 2\ln\left(\frac{w}{p_2}\right) & \text{if } p_1 \geq \frac{w}{2}. \end{cases} \\ &= v(p,w) \end{aligned}$$

To verify that $v(p,w)$ is strictly increasing in w:

$$\frac{\partial v(p,w)}{\partial w} = \begin{cases} \frac{1}{p_1} > 0 & \text{if } p_1 < \frac{w}{2}, \quad \text{and} \\ 2\left(\frac{p_2}{w}\right)\left(\frac{1}{p_2}\right) = \frac{2}{w} > 0 & \text{if } p_1 \geq \frac{w}{2}. \end{cases}$$

So, $v(p,w)$ is strictly increasing in w.

Next,

$$\frac{\partial v(p,w)}{\partial p_2} = \begin{cases} -\frac{2}{p_2} < 0 & \text{if } p_1 < \frac{w}{2}, \quad \text{and} \\ -\frac{2}{p_2} < 0 & \text{if } p_1 \geq \frac{w}{2}. \end{cases}$$

So, $v(p,w)$ is non-increasing (in fact, strictly decreasing) in p_2. To check that $v(p,w)$ is non-increasing in p_1, we differentiate $v(p,w)$ at $p_1 \neq \frac{w}{2}$. Then,

$$\frac{\partial v(p,w)}{\partial p_1} = \begin{cases} -\frac{w}{p_1^2} + \frac{2}{p_1} = \frac{2p_1 - w}{p_1^2} < 0 & \text{if } p_1 < \frac{w}{2}, \quad \text{and} \\ 0 & \text{if } p_1 > \frac{w}{2}. \end{cases}$$

So, $v(p,w)$ is non-increasing everywhere, except may be at $p_1 \neq \frac{w}{2}$. But, since $v(p,w)$ is continuous, it is actually non-increasing everywhere.

7. An indirect utility function $v(p,w)$ is said to have a *Gorman form* if $v(p,w) = a(p) + b(p)w$. Show that the corresponding demand function exhibits linear wealth expansion curves. That is, show that $\frac{\partial x(p,w)}{\partial w}$ is a linear function of w.

Solution: By using Roy's Identity, we obtain

$$x_\ell(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_\ell}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\frac{\partial a(p)}{\partial p_\ell} + \frac{\partial b(p)}{\partial p_\ell} w}{b(p)},$$

which is a linear function of w.

8. Suppose that in a three-goods universe, a consumer's indirect utility function is given by
$$v(p,w) = \left(\frac{1}{2p_1}\right)^{\frac{1}{2}} \left(\frac{1}{8p_2}\right)^{\frac{1}{8}} \left(\frac{3}{8p_3}\right)^{\frac{3}{8}} w.$$

   (a) Find the corresponding Marshallian demand function.

   Solution: For convenience, write $v(p,w)$ as
$$v(p,w) = (2p_1)^{-\frac{1}{2}}(8p_2)^{-\frac{1}{8}}\left(\frac{8p_3}{3}\right)^{-\frac{3}{8}} w.$$

   Applying Roy's Identity yields
$$\begin{aligned}
x_1(p,w) &= -\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\left(-\frac{1}{2}\right)(2)(2p_1)^{-\frac{3}{2}}(8p_2)^{-\frac{1}{8}}\left(\frac{8p_3}{3}\right)^{-\frac{3}{8}} w}{(2p_1)^{-\frac{1}{2}}(8p_2)^{-\frac{1}{8}}\left(\frac{8p_3}{3}\right)^{-\frac{3}{8}}} \\
&= \frac{w}{2p_1}, \\
x_2(p,w) &= -\frac{\frac{\partial v(p,w)}{\partial p_2}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{(2p_1)^{-\frac{1}{2}}\left(-\frac{1}{8}\right)(8)(8p_2)^{-\frac{9}{8}}\left(\frac{8p_3}{3}\right)^{-\frac{3}{8}} w}{(2p_1)^{-\frac{1}{2}}(8p_2)^{-\frac{1}{8}}\left(\frac{8p_3}{3}\right)^{-\frac{3}{8}}} \\
&= \frac{w}{8p_2}, \\
x_3(p,w) &= -\frac{\frac{\partial v(p,w)}{\partial p_3}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{(2p_1)^{-\frac{1}{2}}(8p_2)^{-\frac{1}{8}}\left(-\frac{3}{8}\right)\left(\frac{8}{3}\right)\left(\frac{8p_3}{3}\right)^{-\frac{11}{8}} w}{(2p_1)^{-\frac{1}{2}}(8p_2)^{-\frac{1}{8}}\left(\frac{8p_3}{3}\right)^{-\frac{3}{8}}} \\
&= \frac{3w}{8p_3}.
\end{aligned}$$

   (b) Find the corresponding expenditure function.

   Solution: Apply the duality relationship $v(p,e(p,u)) = u$ yields
$$\begin{aligned}
\left(\frac{1}{2p_1}\right)^{\frac{1}{2}}\left(\frac{1}{8p_2}\right)^{\frac{1}{8}}\left(\frac{3}{8p_3}\right)^{\frac{3}{8}} e(p,u) &= u \\
\Rightarrow \quad e(p,u) &= (2p_1)^{\frac{1}{2}}(8p_2)^{\frac{1}{8}}\left(\frac{8p_3}{3}\right)^{\frac{3}{8}} u
\end{aligned}$$

   (c) Find the corresponding Hicksian demand function.

Solution: Applying the Shepard's Lemma yields:

$$
\begin{aligned}
x_1^h(p,u) &= \frac{\partial e(p,u)}{\partial p_1} = \frac{1}{2}(2)(2p_1)^{-\frac{1}{2}}(8p_2)^{\frac{1}{8}}\left(\frac{8p_3}{3}\right)^{\frac{3}{8}}u \\
&= \left(\frac{4^4 p_2 p_3^3}{3^3 p_1^4}\right)^{\frac{1}{8}} u, \\
x_2^h(p,u) &= \frac{\partial e(p,u)}{\partial p_2} = (2p_1)^{\frac{1}{2}}\left(\frac{1}{8}\right)(8)(8p_2)^{-\frac{7}{8}}\left(\frac{8p_3}{3}\right)^{\frac{3}{8}}u \\
&= \left(\frac{p_1^4 p_3^3}{3^3 4^4 p_2^7}\right)^{\frac{1}{8}} u, \\
x_3^h(p,u) &= \frac{\partial e(p,u)}{\partial p_3} = (2p_1)^{\frac{1}{2}}(8p_2)^{\frac{1}{8}}\left(\frac{3}{8}\right)\left(\frac{8}{3}\right)\left(\frac{8p_3}{3}\right)^{-\frac{5}{8}}u \\
&= \left(\frac{3^5 p_1^4 p_2}{4^4 p_3^5}\right)^{\frac{1}{8}} u.
\end{aligned}
$$

9. Suppose there are L-goods in the economy and a consumer's Marshallian demand function is given by

$$
x_\ell(p,w) = \frac{w}{\sum_{k=1}^{L} p_k} \qquad \forall \ell.
$$

(a) Find the Slutsky matrix $S(p,w)$.

Solution: The row ℓ, column k of the Slutsky matrix is:

$$
\begin{aligned}
S_{\ell k}(p,w) &= \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w) \\
&= \frac{0-w}{\left(\sum_{j=1}^{L} p_j\right)^2} + \left(\frac{1}{\sum_{j=1}^{L} p_j}\right)\left(\frac{w}{\sum_{j=1}^{L} p_j}\right) \\
&= 0.
\end{aligned}
$$

So the Slutsky matrix is the $L \times L$ zero matrix:

$$
S(p,w) = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.
$$

That is, the substitution effects for this demand function are all zero. Can you guess the utility function corresponding to this demand function?

(b) Show whether it is negative semi-definite.

Solution: A zero matrix is negative semi-definite.

(c) Show whether it is negative definite.

Solution: A zero matrix is is not negative definite.

(d) Show whether it is symmetric.

Solution: A zero matrix is symmetric.

10. Consider the following two-period consumption and savings problem. There is a single good which costs p in both periods. The consumer's utility from consuming x_1 amount of the good in period 1 and x_2 amount of the good in period 2 is given by $u(x_1, x_2)$. Assume that $u(\cdot)$ satisfies the standard properties.

The consumer receives wealth W in period 1 and no wealth in period 2. However, any wealth she doesn't spend in period 1 can be saved at interest rate r. That is ¥1 saved in period 1 returns ¥$(1+r)$ in period 2.

(a) What is the budget constraint for the consumer?

Solution: Letting S denote the amount of savings in period 1, we obtain

$$\begin{aligned} px_1 &= W - S \\ px_2 &= (1+r)S \\ \Rightarrow \quad px_1 + \frac{p}{1+r}x_2 &= W \end{aligned}$$

This equation implies that we can treat UMP and EMP in this setup as an ordinary UMP and EMP, with p as the price of good 1 and $\frac{p}{1+r}$ as the price of good 2.

(b) Find the income and the substitution effects of p on x_1.

Solution: The expenditure function $e(p, r, u)$ is the value function for the EMP:

$$\min_{x_1, x_2} px_1 + \frac{p}{1+r}x_2 \quad \text{s.t. } u(x_1, x_2) = u.$$

The associated Lagrangian is

$$\mathscr{L} = px_1 + \frac{p}{1+r}x_2 + \mu[u - u(x)].$$

Applying the envelope theorem, we obtain

$$\frac{\partial e(p,r,u)}{\partial p} = \left.\frac{\partial \mathscr{L}}{\partial p}\right|_{x_1^*, x_2^*, \mu^*} = h_1(p,r,u) + \frac{1}{(1+r)}h_2(p,r,u).$$

Next, since the relative "prices" of good one and good two,

$$\frac{p}{\frac{p}{1+r}} = 1 + r,$$

do not depend on p, $\frac{\partial h_\ell}{\partial p} = 0$. That is, the substitution effect with respect to changes in p is zero.

Differentiating both sides of the identity

$$h_1(p,r,u) = x_1(p,r,e(p,r,u))$$

yields

$$\begin{aligned}
\frac{\partial}{\partial p} h_1(p,r,u) &= \frac{\partial}{\partial p} x_1(p,r,e(p,r,u)) \\
\Rightarrow \quad 0 &= \frac{\partial x_1(p,r,e(p,r,u))}{\partial p} \\
&\quad + \left(\frac{\partial x_1(p,r,e(p,r,u))}{\partial w}\right)\left(\frac{\partial e(p,r,u)}{\partial p}\right) \\
&= \frac{\partial x_1(p,r,W)}{\partial p} + \left(\frac{\partial x_1(p,r,W)}{\partial w}\right)\left(h_1(p,r,u) + \frac{1}{1+r} h_2(p,r,u)\right) \\
\Rightarrow \quad \frac{\partial x_1(p,r,W)}{\partial p} &= -\left(\frac{\partial x_1(p,r,W)}{\partial w}\right)\left(x_1(p,r,W) + \frac{1}{1+r} x_2(p,r,W)\right) \\
&= \underbrace{0}_{\text{SE}} \underbrace{-\left(\frac{\partial x_1(p,r,W)}{\partial w}\right)\left(\frac{W}{p}\right)}_{\text{IE}}.
\end{aligned}$$

(c) Find the income and the substitution effects of r on x_2.

Solution: Using

$$\frac{\partial e(p,r,u)}{\partial r} = \frac{\partial \mathscr{L}}{\partial r}\bigg|_{x_1^*, x_2^*, \mu^*} = \left(\frac{-p}{(1+r)^2}\right) h_2(p,r,u),$$

we obtain

$$\begin{aligned}
\frac{\partial}{\partial r} h_2(p,r,u) &= \frac{\partial}{\partial r} x_2(p,r,e(p,r,u)) \\
&= \frac{\partial x_2(p,r,e(p,r,u))}{\partial r} \\
&\quad + \left(\frac{\partial x_2(p,r,e(p,r,u))}{\partial w}\right)\left(\frac{\partial e(p,r,u)}{\partial r}\right) \\
&= \frac{\partial x_2(p,r,W)}{\partial r} + \left(\frac{\partial x_2(p,r,W)}{\partial w}\right)\left(\frac{-p}{(1+r)^2}\right) h_2(p,r,u) \\
\Rightarrow \quad \frac{\partial x_2(p,r,W)}{\partial r} &= \underbrace{\frac{\partial h_2(p,r,u)}{\partial r}}_{\text{SE}} + \underbrace{\left(\frac{\partial x_2(p,r,W)}{\partial w}\right)\left(\frac{p}{(1+r)^2}\right) x_2(p,r,W)}_{\text{IE}}
\end{aligned}$$

11. Good ℓ is a substitute for good k if $\frac{\partial h_\ell(p,u)}{\partial p_k} \geq 0$. It is a gross substitute if $\frac{\partial x_\ell(p,w)}{\partial p_k} \geq 0$.

(a) Is the property of being substitute a symmetric relationship? That is, is ℓ a substitute for k if and only if k is a substitute for ℓ?

Solution: Yes,

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial h_k(p,u)}{\partial p_\ell}.$$

(b) Suppose good ℓ is a substitute for k. What further restrictions are needed to ensure that it is a gross substitute?

Solution: Slutsky equation yields

$$\frac{\partial x_\ell(p,w)}{\partial p_k} = \underbrace{\frac{\partial h_\ell(p,u)}{\partial p_k}}_{\text{SE }(+)} - \underbrace{x_k(p,w)\frac{\partial x_\ell(p,w)}{\partial w}}_{\text{IE}}.$$

Since SE is positive by assumption, we need good 1 to be either (i) inferior or (ii) normal with $|IE| < |SE|$.

12. A good is said to be Giffen if its own price effect is positive; that is, $\frac{\partial x_i(p,w)}{\partial p_i} > 0$. Let $u(x_1,...,x_N) = x_1 + \phi(x_2,...,x_N)$, where ϕ is concave, be a quasilinear utility function. Show whether any good can be a Giffen good for this utility function.

Solution: The first order conditions for utility maximization problem are:

$$\begin{aligned}
&(1) \quad 1 = \lambda p_1 \\
&(2) \quad \forall i \neq 1, \quad \frac{\partial \phi(x_2^*,...,x_N^*)}{\partial x_i^*} \leq \lambda p_i \quad (= 0 \text{ if } x_i^* > 0) \\
&(3) \quad p \cdot x^* = w,
\end{aligned}$$

which yields

$$\begin{aligned}
&(4) \quad \forall i \neq 1, \quad \frac{\partial \phi(x_2^*,...,x_N^*)}{\partial x_i^*} \leq \frac{p_i}{p_1} \quad (= 0 \text{ if } x_i^* > 0) \\
&(5) \quad x_1^* = \frac{w - p_2 x_2^* - ... - p_N x_n^*}{p_1}
\end{aligned}$$

These first order conditions tell us that w does not affect the optimal consumptions of goods 2,...,N and that all the wealth effect is absorbed by good 1. Therefore, for $i \neq 1$, $\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{p_i} \leq 0$. Since good 1 has to absorb all the wealth effects, this in turns means good 1 must be normal. Therefore, none of the goods can be Giffen.

13. A utility function $u(\cdot)$ is said to be *additively separable* if it has the form $u(x) = u_1(x_1) + ... + u_L(x_L)$.

(a) Show that a linear transformation $\tilde{u}(\cdot) = au(\cdot) + b$, where $a > 0$, of an additively separable utility $u(\cdot)$ is also additively separable.

Solution: Suppose $u(\cdot)$ is additively separable. Then,

$$\begin{aligned}
au(x) + b &= a\left(\sum_{\ell=1}^{L} u_\ell(x_\ell)\right) + b \\
&= \sum_{\ell=1}^{L}\left(au_\ell(x_\ell) + \frac{b}{L}\right)
\end{aligned}$$

(b) Show by example that additive separability need not be preserved if the transformation is merely strictly increasing and not linear.

Solution: Let $u(x) = x_1 + x_2$ and let $f(u) = u^2$. Then $u(x)$ is additively separable but

$$f(u(x)) = (x_1+x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$$

is not additively separable.

(c) Suppose $u(\cdot)$ is additively separable such that $u_\ell(\cdot)$ is strictly concave for all ℓ. Show that none of the goods can be inferior.

Solution: Differentiating the budget identity (Walras Law) with respect to w yields

$$\begin{aligned}\frac{\partial}{\partial w}\left(\sum_{\ell=1}^{L} p_\ell x_\ell(p,w)\right) &= \frac{\partial}{\partial} w \\ \sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p,w)}{\partial w} &= 1.\end{aligned}$$

So, there must be at least one good, say good k, such that $\frac{\partial x_k(p,w)}{\partial w} > 0$. Next, letting $x^* = x(p,w)$ and differentiating the first order condition for UMP with respect to w to yields:

$$\begin{aligned}\frac{\partial}{\partial w}\left(\frac{\partial u(x^*)}{\partial x_\ell}\right) &= \frac{\partial}{\partial w}\left(\frac{p_\ell}{p_k}\frac{\partial u(x^*)}{\partial x_k}\right) \\ \frac{\partial}{\partial w}\left(\frac{\partial u_\ell(x_\ell^*)}{\partial x_\ell}\right) &= \frac{\partial}{\partial w}\left(\frac{p_\ell}{p_k}\frac{\partial u_k(x_k^*)}{\partial x_k}\right) \\ \underbrace{\left(\frac{\partial^2 u_\ell(x_\ell^*)}{\partial x_\ell^2}\right)}_{(-)\text{ by s. concavity}}\left(\frac{\partial x_\ell^*}{\partial w}\right) &= \underbrace{\left(\frac{\partial^2 u_k(x_k^*)}{\partial x_k^2}\right)}_{(-)\text{ by s. concavity}}\underbrace{\left(\frac{\partial x_k^*}{\partial w}\right)}_{(+)\text{ by above}}\end{aligned}$$

Therefore, $\frac{\partial x_\ell^*}{\partial w} > 0$ for all ℓ as required.

14. We say that a utility function is additively separable if $u(x) = \sum_\ell^L u_\ell(x_\ell)$, where $u_\ell(x_\ell)$ is the utility gained from consuming x_ℓ amount of good ℓ.

(a) Must a strictly increasing transformation of an additively separable utility also additively separable? That is, is $\tilde{u}(x) = f(u(x))$, where $f(\cdot)$ is a strictly increasing function necessarily additively separable?

Solution: No. Let $f(u) = u^2$. Then,

$$f(u(x)) = \left(\sum_\ell u_\ell(x_\ell)\right)^2,$$

which is not additively separable.

(b) Suppose $u_\ell(\cdot)$'s are all strictly concave and differentiable. Show whether any good can be an inferior good.

Solution: Since $p \cdot x(p,w) = w$ by Walras' Law, if w increases to w', then the demand for at least one good must increase, call it good k. Letting x^* be the demand when wealth is w and x' the demand when wealth is w', we have

$$\frac{\partial u_\ell}{\partial x_\ell}\Big|_{x'_\ell} = \frac{p_\ell}{p_k}\left(\frac{\partial u_k}{\partial x_k}\Big|_{x'_k}\right) < \frac{p_\ell}{p_k}\left(\frac{\partial u_k}{\partial x_k}\Big|_{x^*_k}\right) = \frac{\partial u_\ell}{\partial x_\ell}\Big|_{x^*_\ell} \quad \Rightarrow \quad x'_\ell > x^*_\ell.$$

where the inequalities follows from the fact that u_ℓ's are all strictly concave. Therefore, demands for every good must increase. I.e., they are all normal goods.

15. Suppose Marshallian demand $x(p,w)$ is homogeneous of degree one with respect to w.

(a) Show that

$$\frac{\partial x_i(p,w)}{\partial w} = \frac{x_i(p,w)}{w}$$

Solution: Differentiating the HD1 identity with respect to α then setting $\alpha = 1$ yields

$$\begin{aligned} \frac{\partial x_i(p,\alpha w)}{\partial \alpha} = \frac{\partial \alpha x_i(p,w)}{\partial \alpha} \quad &\Rightarrow \quad \frac{w x_i(p,\alpha w)}{\partial w} = x_i(p,w) \\ &\Rightarrow \quad \frac{\partial x_i(p,w)}{\partial w} = \frac{x_i(p,w)}{w}. \end{aligned}$$

(b) Show that the law of demand holds for *uncompensated* price changes. That is, show that

$$dp^T D_p x(p,w) dp \leq 0 \quad \text{for all } dp.$$

Solution: Rearranging Slutsky equation yields

$$\begin{aligned} D_p x(p,w) &= S(p,w) - \begin{bmatrix} x_1 \frac{\partial x_1}{\partial w} & x_2 \frac{\partial x_1}{\partial w} \\ x_1 \frac{\partial x_2}{\partial w} & x_2 \frac{\partial x_2}{\partial w} \end{bmatrix} \\ &= S(p,w) - \begin{bmatrix} x_1 \frac{x_1}{w} & x_2 \frac{x_1}{w} \\ x_1 \frac{x_2}{w} & x_2 \frac{x_2}{w} \end{bmatrix} \quad \text{by Part(a).} \end{aligned}$$

Since $S(p,w)$ is negative semi-definite, we only need to show that the matrix representing the wealth effect is positive semi-definite. We have

$$\begin{aligned} dp^T \begin{bmatrix} x_1 \frac{x_1}{w} & x_2 \frac{x_1}{w} \\ x_1 \frac{x_2}{w} & x_2 \frac{x_2}{w} \end{bmatrix} dp &= \begin{bmatrix} \frac{x_1 x_1 dp_1 + x_2 x_1 dp_2}{w} & \frac{x_1 x_2 dp_1 + x_2 x_2 dp_2}{w} \end{bmatrix} dp \\ &= \frac{x_1^2 dp_1^2 + x_2 x_1 dp_2 dp_1 + x_1 x_2 dp_1 dp_2 + x_2^2 dp_2^2}{w} \\ &= \frac{(x_1 dp_1 + x_2 dp_2)^2}{w} > 0. \end{aligned}$$

16. (MWG 3.I.4) Show that if $u(x)$ is quasilinear with respect to the first good (and we fix $p_1 = 1$), then $CV(p^0, p^1, w) = EV(p^0, p^1, w)$ for any (p^0, p^1, w).

    Solution: Typo: This question is (MWG 3.I.5). Indirect utility function for quasilinear utility can be written as $v(p,w) = w + \phi(p)$ (See, for example MWG 3.D.4(b)). Therefore, we have

$$\begin{aligned} v(p^0, w+EV) &= v(p^1, w) \Rightarrow w + EV + \phi(p^0) = w + \phi(p^1) \\ &\Rightarrow EV = \phi(p^1) - \phi(p^0) \\ v(p^1, w-CV) &= v(p^0, w) \Rightarrow w - CV + \phi(p^1) = w + \phi(p^0) \\ &\Rightarrow CV = \phi(p^1) - \phi(p^0). \end{aligned}$$

17. A consumer with utility function $u(x_1, x_2) = x_1 + x_2$ has income w and faces possible changes in the prices from $p^0 = (1,2)$ to $p^1 = (2,1)$.

    (a) Calculate the effect on the consumer's welfare arising from this price change using both the equivalent variation and the compensating variation.

    Solution: The demand and the indirect utility functions for this utility function are:

$$x(p,w) = \begin{cases} \left(0, \frac{w}{p_2}\right) & \text{if } \frac{p_1}{p_2} > 1 \\ \text{any } (x_1, x_2) \text{ on the budget line} & \text{if } \frac{p_1}{p_2} = 1 \\ \left(\frac{w}{p_1}, 0\right) & \text{if } \frac{p_1}{p_2} < 1 \end{cases}$$

$$v(p,w) = \begin{cases} \frac{w}{p_2} & \text{if } \frac{p_1}{p_2} > 1 \\ \frac{w}{p_1} & \text{if } \frac{p_1}{p_2} = 1 \\ \frac{w}{p_1} & \text{if } \frac{p_1}{p_2} < 1 \end{cases}$$

    Since $v(p^0, w) = w = v(p^1, w)$, we have

$$\begin{aligned} v(p^0, w+EV) = v(p^1, w) \quad &\Rightarrow \quad EV = 0, \quad \text{and} \\ v(p^1, w-CV) = v(p^0, w) \quad &\Rightarrow \quad CV = 0. \end{aligned}$$

    (b) Which of the two measures are bigger in this example? What is the main reason for this result?

    Solution: Here, $CV = EV$ because the consumer can achieve exactly the same utility under both price-wealth combinations. Note that it is not the perfect substitutability that is driving the result. Had the utility function been $u(x_1, x_2) = \min\{x_1, x_2\}$, which has zero substitutability, we would still have $CV = EV = 0$.

18. Consider the utility function $u(x_1,x_2) = (x_1)^{\frac{1}{2}} + (x_2)^{\frac{1}{2}}$. For the following, restrict attention to interior solutions.

(a) Derive the consumer's Marshallian demand function and the indirect utility function.

Solution: The first order conditions are:

$$\frac{\partial u(x)}{\partial x_1} = \frac{1}{2}(x_1)^{-\frac{1}{2}} - \lambda p_1 = 0.$$
$$\frac{\partial u(x)}{\partial x_2} = \frac{1}{2}(x_2)^{-\frac{1}{2}} - \lambda p_2 = 0.$$

Dividing yields:

$$\frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = \left(\frac{p_1}{p_2}\right)^2 x_1.$$

Substituting into the budget constraint yields:

$$p_1 x_1 + p_2\left(\frac{p_1}{p_2}\right)^2 x_1 = w.$$

Thus,

$$x_1(p,w) = \frac{w}{p_1 + \frac{p_1^2}{p_2}} = \frac{p_2 w}{p_1p_2 + p_1^2}, \quad x_2(p,w) = \frac{p_1 w}{p_1 p_2 + p_2^2}, \quad \text{and}$$
$$v(p,w) = \left(\frac{p_2 w}{p_1p_2+p_1^2}\right)^{\frac{1}{2}} + \left(\frac{p_1 w}{p_1p_2+p_2^2}\right)^{\frac{1}{2}}.$$

(b) Verify that the demand function is homogeneous of degree zero and satisfies Walra's law.

Solution: For $\alpha > 0$,

$$x_1(\alpha p, \alpha w) = \frac{(\alpha p_2)(\alpha w)}{(\alpha p_1)(\alpha p_2) + (\alpha p_1)^2} = \frac{p_2 w}{p_1p_2 + p_1^2}$$
$$x_2(\alpha p, \alpha w) = \frac{(\alpha p_1)(\alpha w)}{(\alpha p_1)(\alpha p_2) + (\alpha p_2)^2} = \frac{p_1 w}{p_1p_2 + p_2^2}$$
$$p \cdot x(p,w) = p_1\left(\frac{p_2 w}{p_1p_2+p_1^2}\right) + p_2\left(\frac{p_1 w}{p_1p_2+p_2^2}\right) = w.$$

(c) Good i and good j are called *substitutes* if $\frac{\partial x_i^h}{\partial p_j} > 0$ and *complements* if $\frac{\partial x_i^h}{\partial p_j} <$ 0. Using the Slutsky equation, determine whether the two goods in this example are complements or substitutes.

Solution:

$$\begin{aligned}
\frac{\partial x_1^h(p,u)}{\partial p_2} &= \frac{\partial x_1(p,I)}{\partial p_2} + x_2(p,I)\frac{\partial x_1(p,I)}{\partial I} \\
&= \frac{I\left(p_1p_2+p_1^2\right) - p_1p_2 I}{\left(p_1p_2+p_1^2\right)^2} + \frac{p_1 I}{p_1p_2+p_2^2}\left(\frac{p_2}{p_1p_2+p_1^2}\right) \\
&= \frac{I\left(p_1p_2+p_1^2\right) - p_1p_2 I + p_1p_2 I}{\left(p_1p_2+p_1^2\right)^2} \\
&> 0.
\end{aligned}$$

Thus, the goods are substitutes.

19. Consider a consumer who has utility function $u(x_1,x_2) = x_1 + x_2$ and is facing a price change from $p^0 = (1,1)$ to $p^1 = (1,2)$.

    (a) Find the equivalent and the compensating variations measures of welfare change.
    Solution: At p^0 the Marshallian demand is the entire budget line $\{(x_1,x_2) : x_1 + x_2 = w\}$, and the consumer's indirect utility is $v(p,w) = w$. At p^1 the Marshallian demand is the boundary bundle $(w,0)$, and the consumer's indirect utility is $v(p^1,w) = w$. Therefore,

    $$w = v(p^0, w + EV) = v(p^1, w) = w \;\Rightarrow\; EV = 0$$
    $$w = v(p^1, w - CV) = v(p^0, w) = w \;\Rightarrow\; CV = 0$$

    (b) Find the change in the consumer surplus.
    Solution: Since CS is between EV and CV, CS must be zero as well.

    (c) Which of the two measures is larger? Give an intuitive explanation.
    Solution: EV and CV are same in this example because there is no wealth effect.

20. For the following, assume that there are only two goods in the economy.

    (a) Suppose prices change from p^0 to p^1, where $p_1^0 < p_1^1$ and $p_2^0 = p_2^1$. Show whether the equivalent variation or the compensating variation is larger for this price change.
    Solution:

    $$\begin{aligned} EV(p^0,p^1,w) &= \int_{p_1^1}^{p_1^0} h_1(p_1,p_{-1},u^1)dp_1 \\ CV(p^0,p^1,w) &= \int_{p_1^1}^{p_1^0} h_1(p_1,p_{-1},u^0)dp_1 \end{aligned}$$

    Since $p^0 < p^1$, $u^0 = v(p^0,w) > v(p^1,w) = u^1$. Therefore, $e(p_1,p_2^0,u^0) > e(p_1,p_2^0,u^1)$ for all p_1. If good 1 is normal, this implies that

    $$\begin{aligned} & x_1(p_1,p_2^0,e(p_1,p_2^0,u^0)) > x_1(p_1,p_2^0,e(p_1,p_2^0,u^1)) \\ \Rightarrow\; & h_1(p_1,p_2^0,e(p_1,p_2^0,u^0)) > h_1(p_1,p_2^0,e(p_1,p_2^0,u^1)) \\ & \Rightarrow\; EV > CV \quad \text{since } p_1^0 < p_1^1. \end{aligned}$$

    If good 1 is inferior, this implies that

    $$\begin{aligned} & x_1(p_1,p_2^0,e(p_1,p_2^0,u^0)) < x_1(p_1,p_2^0,e(p_1,p_2^0,u^1)) \\ \Rightarrow\; & h_1(p_1,p_2^0,e(p_1,p_2^0,u^0)) < h_1(p_1,p_2^0,e(p_1,p_2^0,u^1)) \\ & \Rightarrow\; EV < CV \quad \text{since } p_1^0 < p_1^1. \end{aligned}$$

(b) Now, suppose the prices of both goods change. That is, prices change from (p_1^0, p_2^0) to (p_1^1, p_2^1), where $p_1^0 \neq p_1^1$ and $p_2^0 \neq p_2^1$. Express the equivalent variation and the compensating variation in terms of the appropriate Hicksian demands.

Solution:

$$\begin{aligned}
EV(p^0,p^1,w) &= e(p^0,u^1)-e(p^0,u^0) \;=\; e(p^0,u^1)-e(p^1,u^1) \\
&= e(p_1^0,p_2^0,u^1)-e(p_1^1,p_2^0,u^1)+e(p_1^1,p_2^0,u^1)-e(p_1^1,p_2^1,u^1) \\
&= \int_{p_1^1}^{p_1^0} h_1(p_1,p_2^0,u^1)dp_1+\int_{p_2^1}^{p_2^0} h_1(p_1^1,p_2,u^1)dp_2. \\
CV(p^0,p^1,w) &= e(p^1,u^1)-e(p^1,u^0) \;=\; e(p^0,u^0)-e(p^1,u^0) \\
&= e(p_1^0,p_2^0,u^0)-e(p_1^1,p_2^0,u^0)+e(p_1^1,p_2^0,u^0)-e(p_1^1,p_2^1,u^0) \\
&= \int_{p_1^1}^{p_1^0} h_1(p_1,p_2^0,u^0)dp_1+\int_{p_2^1}^{p_2^0} h_1(p_1^1,p_2,u^0)dp_2.
\end{aligned}$$

21. In the following, let $EV(p^0,p^1,w)$ and $EV(p^0,p^2,w)$ denote the equivalent variation measure of welfare change between (p^0,w) and (p^1,w) and between (p^0,w) and (p^2,w), respectively. Let $CV(p^0,p^1,w)$ and $CV(p^0,p^2,w)$ denote the analogous for compensating variation measure of welfare change. Note that we are considering cases in which the individual's wealth does not change.

(a) Show that the equivalent variation measure gives a correct welfare ranking of p^1 versus p^2. That is, $EV(p^0,p^1,w) > EV(p^0,p^2,w)$ if and only if $v(p^1,w) > v(p^2,w)$.

Solution: We have

$$\begin{aligned}
& EV(p^0,p^1,w) > EV(p^0,p^2,w) \\
\iff & e(p^0,v(p^1,w))-e(p^0,v(p^0,w)) > e(p^0,v(p^2,w))-e(p^0,v(p^0,w)) \\
\iff & e(p^0,v(p^1,w)) > e(p^0,v(p^2,w)) \\
\iff & v(p^1,w) > v(p^2,w) \text{ since } e(p^0,u) \text{ is increasing in } u.
\end{aligned}$$

The steps (b)-(d) below construct an example where $CV(p^0,p^1,w)$ and $CV(p^0,p^1,w)$ do not give a correct welfare ranking of p^1 versus p^2. First, let $u(x) = x_1 + \phi(x_2)$, where $\phi(\cdot)$ is increasing and strictly concave. This utility function is an example of a quasilinear utility function, which is linear in one of the goods (x_1 in this case). In the following, assume that $\phi(\cdot)$ is differentiable and that the Marshallian demand will be interior (i.e., $x(p,w) \gg 0$).

(b) Show that good 1 is a normal good and that the wealth effect on good 2 (the nonlinear part of the quasilinear utility) is zero (i.e., $\frac{\partial x_2(p,w)}{\partial w} = 0$).

Solution: Assuming interior solution, the first order condition for the utility maximization problem is the "MRS = price ratio" condition:

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{1}{\phi'(x_2)} = \frac{p_1}{p_2} \implies \phi'(x_2) = \frac{p_2}{p_1}.$$

Since the last equation doesn't involve x_1, $x_2(p,w)$ can be found by solving the equation for x_2 (see for example, PS1, Q1). Thus, we have

$$x_2(p,w) = (\phi')^{-1}\left(\frac{p_2}{p_1}\right) \implies \frac{\partial x_2(p,w)}{\partial w} = 0 \text{ since } x_2(p,w) \text{ does not depend on } w$$
$$x_1(p,w) = \frac{w - p_2 x_2(p,w)}{p_1} \implies \frac{\partial x_1(p,w)}{\partial w} = \frac{1}{p_1} > 0.$$

(c) Let $p^0 = (p_1^0, p_2^0)$, and obtain p^1 from p^0 by lowering the price of good 1 slightly. That is, $p^1 = (p_1^1, p_2^0)$ for some $p_1^1 < p_1^0$. Next, obtain p^2 from p^0 by lowering the price of good 2 until $v(p^1,w) = v(p^2,w)$. That is, $p^2 = (p_1^0, p_2^2)$, where $p_2^2 < p_2^0$ and $v(p^1,w) = v(p^2,w)$. Show that $EV(p^0,p^1,w) = EV(p^0,p^2,w)$, meaning EV ranks p^1 and p^2 correctly.

Solution: Since $v(p^1,w) = v(p^2,w)$ by construction, we have

$$\begin{aligned} EV(p^0,p^1,w) &= e(p^0, v(p^1,w)) - e(p^0, v(p^0,w)) \\ &= e(p^0, v(p^2,w)) - e(p^0, v(p^0,w)) = EV(p^0,p^2,w). \end{aligned}$$

Note in particular this means that EV correctly determines that the individual likes p^1 and p^2 equally.

(d) Show that $CV(p^0,p^1,w) < CV(p^0,p^2,w)$, meaning CV ranks p^1 and p^2 incorrectly.

Solution: As shown in the lecture, we have $p_1^1 < p_1^0$ and good 1 is normal. Therefore, $CV(p^0,p^1,w) < EV(p^0,p^1,w)$. Moreover, since the wealth effect on good 2 is zero, $CV(p^0,p^2,w) = EV(p^0,p^2,w)$. Thus, we have

$$CV(p^0,p^1,w) < EV(p^0,p^1,w) = EV(p^0,p^2,w) = CV(p^0,p^2,w).$$

2 Aggregate Demand

1. MWG 4.D.6

   Solution: Since $u(\cdot)$ is homothetic, its indirect utility function $v(p,w)$ is HD1 in w. (See, for example MWG 3.D.3(a)). Therefore,

$$v_i(p,\gamma w_i) = \gamma v_i(p,w_i) \quad \Rightarrow \quad \frac{\partial v(p,\gamma w_i)}{\partial \gamma}\Big|_{\gamma=1} = \frac{\partial \gamma v(p,w_i)}{\partial \gamma}\Big|_{\gamma=1}$$

$$\Rightarrow \frac{\partial v(p,\gamma w_i)}{\partial w_i}\Big|_{\gamma=1} w_i = v(p,w_i) \quad \Rightarrow \quad \frac{\partial v(p,w_i)}{\partial w_i} = \frac{v(p,w_i)}{w_i}.$$

   Now, consider the social welfare maximization problem and the associated Lagrangian:

$$\max_{w_1,\ldots,w_I} \sum_i \alpha_i \ln v_i(p,w_i) \quad \text{s.t.} \quad \sum_i w_i = w$$

$$\mathscr{L} = \sum_i \alpha_i \ln v_i(p,w_i) + \lambda \left[w - \sum_i w_i \right].$$

   The first order condition is, for all i,

$$\frac{\partial \mathscr{L}}{\partial w_i} = \frac{\alpha_i}{v_i(p,w_i)} \frac{\partial v_i(p,w_i)}{\partial w_i} - \lambda = 0.$$

   Using, above, we obtain

$$\begin{aligned} \frac{\alpha_i}{v_i(p,w_i)} \frac{v_i(p,w_i)}{w_i} &= \lambda \\ \Rightarrow \quad w_i &= \frac{\alpha_i}{\lambda} \\ \Rightarrow \quad w = \sum_i w_i &= \frac{\sum_i \alpha_i}{\lambda} = \frac{1}{\lambda} \\ \Rightarrow \quad w_i &= \alpha_i w. \end{aligned}$$

2. Suppose there are two consumers with identical utility function

$$u_i(x_{1i}, x_{2i}) = x_{1i} + (x_{2i})^{\frac{1}{2}}$$

   where $x_{\ell i}$ denotes good ℓ for consumer i, and identical wealth $w_i = \frac{w}{2}$. For the following, restrict attention to interior solutions only.

   (a) Show that the aggregate demand function can be expressed as a function of price and aggregate wealth.

   Solution: Interior first order conditions for utility maximization yield

$$\frac{\frac{\partial u_i}{\partial x_{1i}}}{\frac{\partial u_i}{\partial x_{2i}}} = \frac{1}{\frac{1}{2} x_{2i}^{-\frac{1}{2}}} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_{2i} = \frac{p_1^2}{4p_2^2}$$

$$p_1 x_{1i} + p_2 x_{2i} = w_i \quad \Rightarrow \quad x_{1i} = \frac{w_i}{p_1} - \frac{p_1}{4p_2}.$$

The wealth expansion paths of the two consumers are clearly parallel lines: $\frac{\partial x_{1i}}{\partial w} = \frac{1}{p_1}$ and $\frac{\partial x_{2i}}{\partial w} = 0$. Thus, aggregate demand can be expressed as a function of prices and aggregate wealth.

(b) Find the aggregate demand function, $x(p,w)$.

Solution: Summing the demand functions of the two consumers yields

$$x(p,w) = \left(\frac{w}{p_1} - \frac{p_1}{2p_2} , \frac{p_1^2}{2p_2^2} \right).$$

(c) Show whether the aggregate demand satisfies the compensated law of demand.

Solution: One can calculate the Slutsky matrix $S(p,w)$ and show that it is negative semi-definite, or recognize that $x(p,w)$ is a demand function for quasi-linear utility

$$u(x_1,x_2) = x_1 + \left(\frac{1}{\sqrt{2}}\right) x_2^{\frac{1}{2}}.$$