Advanced Microeconomics I

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Midterm: Suggested Solutions

1. [20] An individual's indirect utility function is given by

$$v(p,w) = \frac{4w^2}{(2p_1+p_2)^2}.$$

   (a) [6] Find the Marshallian demand.
   Solution: Using Roy's identity on $v(p,w) = 4w^2(2p_1+p_2)^{-2}$, we obtain

$$x_1(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{-8w^2(2p_1+p_2)^{-3}(2)}{8w(2p_1+p_2)^{-2}} = \frac{2w}{2p_1+p_2}$$

$$x_2(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_2}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{-8w^2(2p_1+p_2)^{-3}}{8w(2p_1+p_2)^{-2}} = \frac{w}{2p_1+p_2}.$$

   (b) [5] Find the expenditure function.
   Solution: Since $v(p,e(p,u)) = u$, we have

$$\frac{4e(p,u)^2}{(2p_1+p_2)^2} = u \iff e(p,u) = \frac{(2p_1+p_2)u^{\frac{1}{2}}}{2}.$$

   (c) [5] Find the Hicksian demand.
   Solution: Shepard's lemma yields,

$$h_1(p,u) = \frac{\partial e(p,u)}{\partial p_1} = u^{\frac{1}{2}} \quad \text{and} \quad h_2(p,u) = \frac{\partial e(p,u)}{\partial p_2} = \frac{u^{\frac{1}{2}}}{2}.$$

   (d) [4] Find the substitution effects for the two goods. What does this say about the class of (direct) utility function the individual has (for example, is it Cobb-Douglas)?
   Solution: Since the Hicksian demands are not functions of prices, the substitution effect is zero (that is, $\frac{\partial x_l^h}{\partial p_k} = 0$). This implies that the goods are perfect complements to the individual (i.e., Leontieff utility function).

2. [20] An individual has a quasilinear utility function $u(x_1, x_2) = x_1 + \phi(x_2)$, where $\phi' > 0$ and $\phi'' < 0$. For this question, ignore boundary solutions and assume that the Marshallian demand is interior (that is, $x(p, w) \gg 0$).

   (a) [4] Carefully graph the wealth expansion paths for this individual. Use the horizontal axis for good 1 and the vertical axis for good 2.

   Solution: Letting $\phi' = \frac{\partial \phi}{\partial x_2}$, the first order conditions for UMP are:

$$\frac{\partial u(x_1, x_2)}{\partial x_1} = \lambda p_1 \implies 1 = \lambda p_1$$
$$\frac{\partial u(x_1, x_2)}{\partial x_2} = \lambda p_2 \implies \phi'(x_2) = \lambda p_2 \implies \phi'(x_2) = \frac{p_2}{p_1}.$$

   Note that $\phi'(x_2)$ is the inverse demand function of the individual, and her demand function for good 2, $x_2(p, w)$, is the inverse function of ϕ' and does not depend on w (that is, $\frac{\partial x_2}{\partial w} = 0$). Thus, the wealth expansion paths are horizontal lines (on a graph with good 1 on the horizontal axis and good 2 on the vertical axis).

   (b) [2] Explain the circumstances under which a quasilinear utility may be a reasonable model of an individual's preference.

   Solution: As see above, a quasilinear utility means that the demand for the non-numeraire good is independent of income. Therefore it is arguably reasonable for modeling a good like a postage stamp that is unlikely to increase once income goes above some minimal level.

   (c) [8] Suppose the demand for good 2 is given by $x_2(p_2) = p_1^2/p_2^2$. Find the substitution and the income effects on **good 1** arising from a change in the price of good 2.

   Solution: From the budget equation $p_1 x_1 + p_2 x_2 = w$, we have

$$x_1(p, w) = \frac{w - p_2 x_2}{p_1} = \frac{w - p_2 \frac{p_1^2}{p_2^2}}{p_1} = \frac{w}{p_1} - \frac{p_1}{p_2}.$$

   Thus, using Slutsky equation, we obtain

$$S_{12}(p, w) = \frac{\partial x_1(p, w)}{\partial p_2} + x_2(p, w) \frac{\partial x_1(p, w)}{\partial w} = \frac{p_1}{p_2^2} + \frac{p_1^2}{p_2^2}\left(\frac{1}{p_1}\right) = \frac{2p_1}{p_2^2}.$$

   The income effect is:

$$-x_2(p, w) \frac{\partial x_1(p, w)}{\partial w} = -\frac{p_1}{p_2^2}.$$

   (d) [6] Continuing to assume that $x_2(p_2) = p_1^2/p_2^2$, find the equivalent variation measure and the compensating variation measures of welfare change when the price of good 2 increases from 2 RMB to 3 RMB while the price of good 1 remains at 1 RMB.

   Solution: Since the wealth effect is zero for good 2 for this utility function, we have

$$CV = EV = CS = -\int_2^3 x_2(p_2) dp_2 = -\int_2^3 p_2^{-2} dp_2 = p_2^{-1}\Big|_2^3 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}.$$

3. [20] An economy consists of two individuals whose utility functions are:

$$u_1(x_{11}, x_{21}) = \frac{x_{11}^{\frac{2}{3}} x_{21}^{\frac{1}{3}}}{(\frac{2}{3})^{\frac{2}{3}}(\frac{1}{3})^{\frac{1}{3}}} \quad \text{and} \quad u_2(x_{12}, x_{22}) = \frac{x_{12}^{\frac{1}{3}} x_{22}^{\frac{2}{3}}}{(\frac{1}{3})^{\frac{1}{3}}(\frac{2}{3})^{\frac{2}{3}}},$$

where $x_{\ell i}$ denotes the amount of good ℓ for individual i. Let w be the aggregate wealth. Suppose the central planner redistributes wealth to maximize the social welfare function $W(u_1, u_2) = \min\{u_1, u_2\}$.

(a) [10] Find the optimal allocation of wealth. When will individual 1 receive more than half of the aggregate wealth. (You do not need to derive the Marshallian demand for a Cobb-Douglas utility function if you know what it is).

Solution Using the formula for Cobb-Douglas utility function, we have (subscript refers to individuals not goods):

$$x_1(p, w_1) = \left(\frac{2w_1}{3p_1}, \frac{w_1}{3p_2}\right) \Longrightarrow v_1(p, w_1) = \frac{(\frac{2}{3})^{\frac{2}{3}}(\frac{1}{3})^{\frac{1}{3}} w_1}{(\frac{2}{3})^{\frac{2}{3}}(\frac{1}{3})^{\frac{1}{3}} p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}} = \frac{w_1}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}$$

$$x_2(p, w_2) = \left(\frac{w_2}{3p_1}, \frac{2w_2}{3p_2}\right) \Longrightarrow v_2(p, w_2) = \frac{(\frac{1}{3})^{\frac{1}{3}}(\frac{2}{3})^{\frac{2}{3}} w_2}{(\frac{1}{3})^{\frac{1}{3}}(\frac{2}{3})^{\frac{2}{3}} p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}} = \frac{w_2}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}.$$

$$\text{Central planner problem:} \quad \max_{w_1, w_2} \left(\min\left\{\frac{w_1}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}, \frac{w_2}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}\right\}\right) \quad \text{s.t.} \quad w_1 + w_2 = w$$

$$\Longrightarrow \frac{w_1}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}} = \frac{w - w_1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}} \Longleftrightarrow p_2^{\frac{1}{3}} w_1 = p_1^{\frac{1}{3}}(w - w_1) \Longrightarrow w_1 = \frac{p_1^{\frac{1}{3}} w}{p_1^{\frac{1}{3}} + p_2^{\frac{1}{3}}} \ \& \ w_2 = \frac{p_2^{\frac{1}{3}} w}{p_1^{\frac{1}{3}} + p_2^{\frac{1}{3}}}.$$

Thus, $w_1 > w_2 \Longleftrightarrow p_1 > p_2$.

(b) [4] Find the indirect utility function of the representative individual.

Solution: From (a), the indirect utility function of the representative individual is

$$V(p, w) = \frac{\frac{p_1^{\frac{1}{3}} w}{p_1^{\frac{1}{3}} + p_2^{\frac{1}{3}}}}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}} = \frac{w}{p_1^{\frac{1}{3}} p_2^{\frac{1}{3}} (p_1^{\frac{1}{3}} + p_2^{\frac{1}{3}})}.$$

(c) [6] Suppose the central planner is considering changing the prices in the economy from $p^0 = (8, 64)$ to $p^1 = (27, 8)$. Find the equivalent variation and the compensating variation measures of the welfare change on the representative individual. Should the central planner make the change?

Solution: Using $V(p^0, w + EV) = V(p^1, w)$, we have

$$V(p^0, w^0) = \frac{w^0}{p_1^{\frac{1}{3}} p_2^{\frac{1}{3}} (p_1^{\frac{1}{3}} + p_2^{\frac{1}{3}})} = \frac{w^0}{8^{\frac{1}{3}} 64^{\frac{1}{3}} (8^{\frac{1}{3}} + 64^{\frac{1}{3}})} = \frac{w^0}{2(4)(2+4)} = \frac{w^0}{48}$$

$$V(p^1, w^1) = \frac{w^1}{p_1^{\frac{1}{3}} p_2^{\frac{1}{3}} (p_1^{\frac{1}{3}} + p_2^{\frac{1}{3}})} = \frac{w^1}{27^{\frac{1}{3}} 8^{\frac{1}{3}} (27^{\frac{1}{3}} + 8^{\frac{1}{3}})} = \frac{w^1}{3(2)(3+2)} = \frac{w^1}{30}$$

$$V(p^0, w^0 + EV) = V(p^1, w^0) \implies \frac{w + EV}{48} = \frac{w}{30} \iff w + EV = \frac{8w}{5} \implies EV = \frac{3w}{5}$$

$$V(p^1, w^1 - CV) = V(p^0, w^0) \implies \frac{w - CV}{30} = \frac{w}{48} \iff w - CV = \frac{18w}{48} \implies CV = \frac{3w}{8}$$

4. [20] Suppose a firm's (single-output) production function is $f(z) = z_1^b + z_2^b$, where $b > 0$. In the following, let $q > 0$ denote the output quantity.

(a) [8] Find all the values of b for which the firm's production function is quasiconcave (that is, its upper contour sets, $\{z \in \mathbb{R}_+^2 : f(z) \geq q\}$, are convex sets).

Solution: Let $z = (z_1, z_2)$ and $z' = (z_1', z_2')$ be such that $f(z) = f(z')$. Let $z'' = \alpha z + (1-\alpha)z'$ for some $0 < \alpha < 1$. First, note that $g(z_\ell) = z_\ell^b$ is concave if and only if $0 \leq b \leq 1$. That is, $\left(\alpha z_\ell + (1-\alpha)z_\ell'\right)^b > \alpha z_\ell^b + (1-\alpha)z_\ell'^b$ if and only if $0 \leq b \leq 1$. Next, note that we need $f(z'') \geq f(z)$ for quasiconcavity. This means,

$$\begin{aligned} f(z'') &= \left(\alpha z_1 + (1-\alpha)z_1'\right)^b + \left(\alpha z_2 + (1-\alpha)z_2'\right)^b \\ &\geq \alpha z_1^b + (1-\alpha)z_1'^b + \alpha z_2^b + (1-\alpha)z_2'^b \quad \text{if and only if } 0 \leq b \leq 1 \\ &= \alpha f(z) + (1-\alpha)f(z') = \alpha f(z) + (1-\alpha)f(z) = f(z). \end{aligned}$$

Therefore, we need $0 \leq b \leq 1$.

(b) [4] Find all the values of b for which the firm's production is strictly concave.

Solution: We have $\frac{\partial f}{\partial z_\ell} = b z_\ell^{b-1}$. Thus, for all $x \neq 0$,

$$\begin{aligned} x^T H_f x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} b(b-1)z_1^{b-2} & 0 \\ 0 & b(b-1)z_1^{b-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= b(b-1)x_1^2 z_1^{b-2} + b(b-1)x_1^2 z_2^{b-2} < 0 \iff 0 < b < 1. \end{aligned}$$

(c) [8] Assuming that the firm's production function is strictly concave, find the firm's cost function and show that it is a strictly convex function of the output level q.

Solution:

$$\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} = \frac{b z_1^{b-1}}{b z_2^{b-1}} = \frac{w_1}{w_2} \iff \frac{z_2^{1-b}}{z_1^{1-b}} = \frac{w_1}{w_2} \iff z_2 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{1-b}} z_1.$$

$$z_1^b + \left(\frac{w_1}{w_2}\right)^{\frac{b}{1-b}} z_1^b = q \implies z_1^b = \left(\frac{w_1^{\frac{b}{1-b}} + w_2^{\frac{b}{1-b}}}{w_2^{\frac{b}{1-b}}}\right)^{-1} q \implies z_1 = \left(\frac{w_2^{\frac{b}{1-b}}}{w_1^{\frac{b}{1-b}} + w_2^{\frac{b}{1-b}}}\right)^{\frac{1}{b}} q^{\frac{1}{b}}.$$

$$c(w,q) = w_1 \left(\frac{w_2^{\frac{b}{1-b}}}{w_1^{\frac{b}{1-b}} + w_2^{\frac{b}{1-b}}}\right)^{\frac{1}{b}} q^{\frac{1}{b}} + w_2 \left(\frac{w_1^{\frac{b}{1-b}}}{w_1^{\frac{b}{1-b}} + w_2^{\frac{b}{1-b}}}\right)^{\frac{1}{b}} q^{\frac{1}{b}},$$

which is strictly concave since $\frac{1}{b} > 1$.

5. [20] A profit maximizing firm uses labor and capital to produce an output. The firm's profit function is

$$\pi(p,w,r) = \frac{p(p-w-r)}{2(w+r)},$$

where p, w, and r are the prices of the output good, labor, and capital, respectively, and $p > w+r$.

(a) [8] Find the firm's supply function and the unconditional input demands.

Solution: [4] Using Hotelling's Lemma and Shepard's Lemma, we obtain:

$$y(p,w,r) = \frac{\partial\pi(p,w,r)}{\partial p} = \frac{p-w-r+p}{2(w+r)} = \frac{2p-w-r}{2(w+r)} = \frac{p}{w+r} - \frac{1}{2}$$

$$z_L(p,w,r) = -\frac{\partial\pi(p,w,r)}{\partial w} = -\frac{-p2(w+r)-p(p-w-r)2}{4(w+r)^2}$$

$$= -\frac{-2pw-2pr-2p^2+2pw+2pr}{4(w+r)^2} = \frac{p^2}{2(w+r)^2} = z_K(p,w,r),$$

where the last equality for $z_K(p,w,r)$ follows by symmetry.

(b) [4] What degree of homogeneity in prices, if any, do the supply and the profit functions of this firm exhibit? Show whether this is true in general (that is, not just for this firm).

Solution: The supply function is HD0 and the profit function is HD1.

$$y(\alpha p,\alpha w,\alpha r) = \frac{(\alpha p)^2}{2(\alpha w+\alpha r)^2} = \frac{p^2}{2(w+r)^2} = y(p,w,r) \implies HD0$$

$$\pi(\alpha p,\alpha w,\alpha r) = \frac{\alpha p(\alpha p-\alpha w-\alpha r)}{2(\alpha w+\alpha r)} = \frac{\alpha p(p-w-r)}{2(w+r)} = \alpha\pi(p,w,r) \implies HD1.$$

This is true in general because the firm's profit maximization problem is

$$\max_z pf(z) - w\cdot z \iff \max_z \alpha pf(z) - \alpha w\cdot z \text{ for all } \alpha > 0.$$

means that the solution is HD0 and the value function is HD1.

(c) [8] Suppose the government wants to tax the firm to raise money for itself. It can either place a tax rate $0<t<1$ fraction on the output price (that is, the tax collected is tp per unit of output) or place it on the profit of the firm (that is, tax collected is $t\pi$). Show which is better for the firm and which is better for the government.

Solution Let π_p and π_π be the firm's profit when the tax is on the output price and the profit, respectively. The firm always prefers to be taxed on the profit because

$$\pi_\pi = \frac{(1-t)p(p-w-r)}{2(w+r)} > \frac{(1-t)p((1-t)p-w-r)}{2(w+r)} = \pi_p \iff 0 > (1-t)p(-tp),$$

which is always true. Let y_p, T_p, and T_π be the output and the tax revenues when the output price is taxed and the profit is taxed, respectively. The government prefers to tax profit if and only if

$$T_\pi = t\pi_\pi > tpy_p = T_p \iff \frac{tp(p-w-r)}{2(w+r)} > \frac{tp(2(1-t)p-w-r)}{2(w+r)}$$

$$\iff tp^2 > 2tp^2 - 2t^2p^2 \iff 2t^2p^2 > tp^2 \iff t > \frac{1}{2}.$$