PRODUCTION

FIRM ≡ ENTITY THAT TRANSFORMS INPUTS (FACTORS OF PRODUCTION) INTO OUTPUTS USING SOME PRODUCTION PLAN $y \in \mathbb{R}^L$

$Y \subseteq \mathbb{R}^L$ ≡ SET OF ALL POSSIBLE PRODUCTION PLANS THAT A FIRM MAY USE

INTERPRET NEGATIVE COMPONENTS OF $y \in Y$ AS INPUTS & POSITIVE COMPONENTS OF y AS OUTPUTS

EG. $y = (-1, 2, 0, 1, -4)$ = PRODUCTION PLAN y USES 1 UNIT OF GOOD 1 & 4 UNITS OF GOOD 5 TO PRODUCE 2 UNITS OF GOOD 2 & 1 UNIT OF GOOD 4. GOOD 3 IS NEITHER USED NOR CREATED.

y ≡ NETPUT VECTOR

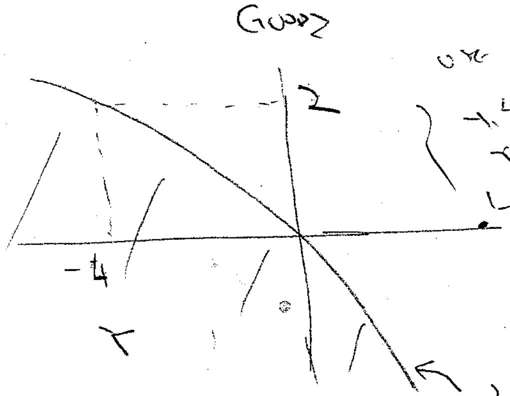
NOTE: THIS CONVENTION MAKES IT EASY TO EXPRESS PROFIT ASSOCIATED WITH A PRODUCTION PLAN. EG GIVEN $p = (p_1, p_2, \ldots, p_5)$,

MOVE TO (A)

$$p \cdot y = (-1)p_1 + (2)p_2 + 0(p_3) + (1)p_3 + (-4)p_5$$

$$= 2p_2 + p_3 - p_1 - 4p_5 \text{ = PROFIT EARNED FROM } y.$$

PICTURE:



USE $\{(y_1, y_2)$: $y_1 \le 0$, $y_2 \le (-y_1)^{1/2}\}$

$(-4, 2) \in Y$

$(-4, 1) \in Y$

$(-4, 3) \notin Y$

$\{y : F(y) = 0\}$ ≡ TRANSFORMATION FRONTIER ≡ PRODUCTION POSSIBILITY FRONTIER

SOME COMMON ASSUMPTION ON Y

① Y IS CLOSED. IE. Y CONTAINS ITS BOUNDARY POINTS. $y_n \to y$ & $y_n \in Y \ \forall n \Rightarrow y \in Y$

② FREE DISPOSAL. IF $y \in Y$ & $y' \leq y$, THEN $y' \in Y$.

③ Y IS CONVEX. (GET. Y CONVEX $\Rightarrow$ DIC RSCALE BALANCED INPUT IS AT LEAST AS GOOD AS EXTREME INPUT.)

SEE MWG 5.B FOR MORE.

DEFN: TRANSFORMATION FUNCTION FOR Y

IS A FUNCTION $F: \mathbb{R}^L \to \mathbb{R}$ S.T.

① $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$

& ② $F(y) = 0$ IFF y IS ON THE BOUNDARY OF Y

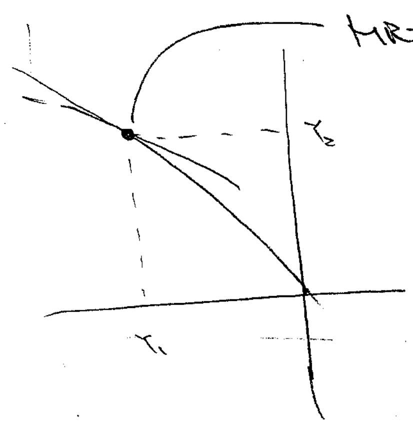
$Y = \{(y_1, y_2) : y_2 \leq (-y_1)^{1/2}$ & $y_1 \leq 0\}$ $\langle F(y) = y_2 - (-y_1)^{1/2} \rangle$

DEFN: FOR y S.T. $F(y) = 0$, DEFIN

MARGINAL TRANSFORMATION OF GOOD l FOR k

$$= MRT_{lk} = \frac{\frac{\partial F(y)}{\partial y_l}}{\frac{\partial F(y)}{\partial y_k}}$$

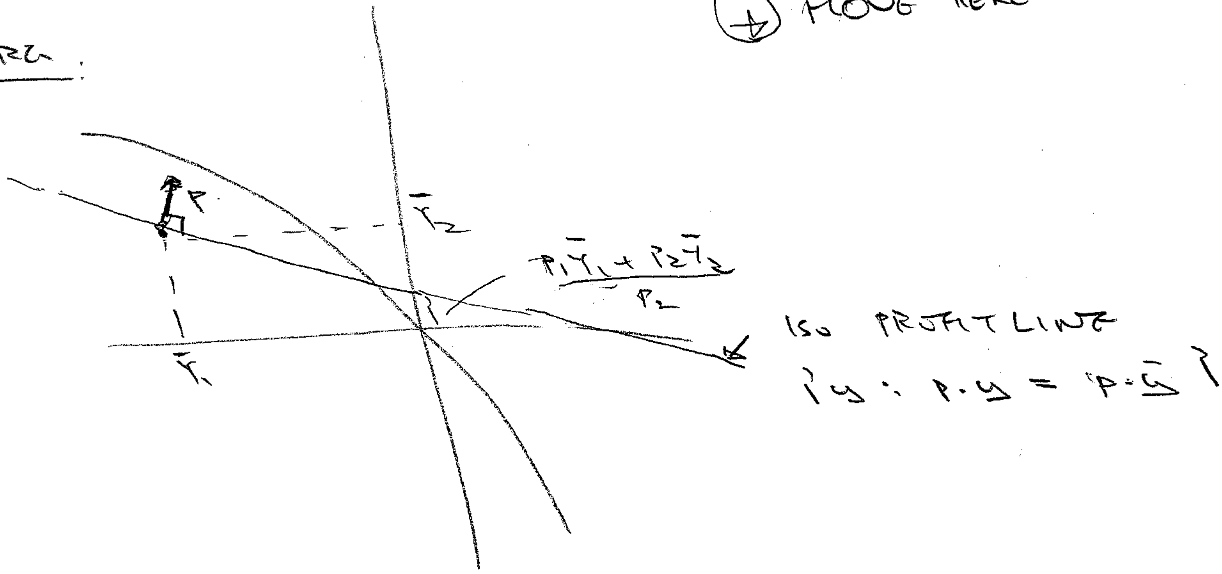
IN TWO GOODS CASE



$$MRT_{12} = \frac{\frac{\partial F(y)}{\partial y_1}}{\frac{\partial F(y)}{\partial y_2}} = - \text{SLOPE OF TRANSFORMATION FRONTIER}$$

(+) MOVE HERE

PICTURE:



FIRM'S OBJECTIVE: MAXIMIZE PROFIT.

πMP: $\max_{y \in Y} \; p \cdot y$

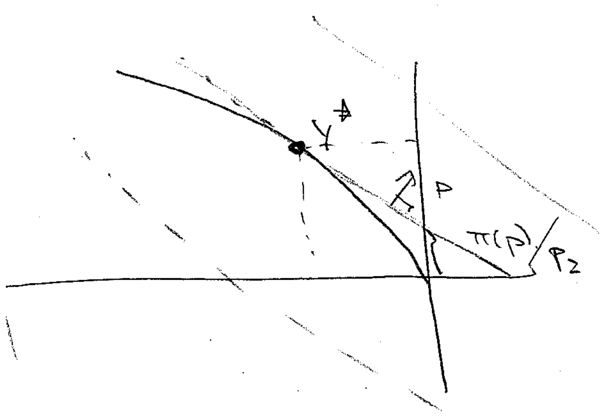
$\Leftrightarrow \max \; p \cdot y \quad$ s.t. $F(y) \leq 0$

To SOLVE: $\mathcal{L} = p \cdot y + \lambda[-F(y)]$ <DIFF. CASE>

FOC: $p_e - \lambda \frac{\partial F(y)}{\partial y_e} = 0 \quad \forall e$ AT INTERIOR

SOC: CAN BE IGNORED IF Y IS CONVEX

PICTURE:



$\Rightarrow$ SOLUTION $y(p)$ = (NET) SUPPLY CORRESPONDENCE

VALUE FUNC. $\pi(p) = p \cdot y(p)$ = PROFIT FUNCTION.

THM. SUPPOSE Y IS CLOSED & SATISFIES FREE DISPOSAL THEN

1) $y(\cdot)$ IS HD0

2) IF Y IS CONVEX THEN $y(\cdot)$ IS CONVEX.
IF Y IS S. CONVEX THEN $y(\cdot)$ IS A SINGLETON (IF NON-EMPTY).

3) $\pi(\cdot)$ IS HD1

4) $\pi(\cdot)$ IS CONVEX

5) HOTELLING'S LEMMA: IF $y(p)$ IS A SINGLETON, THEN $y_\ell(p) = \frac{\partial \pi(p)}{\partial p_\ell}$ $\forall \ell$.

6) IF $y(\cdot)$ IS DIFFERENTIABLE, THEN $Dy(\bar{p}) = D^2\pi(\bar{p})$ IS SYMMETRIC, P.S.D., & SATISFIES $Dy(p)\,p = 0$.

C.F. SHEPARD'S LEMMA & expenditure function

REMARK: $Dy(p)$ IS POSITIVE SEMI-DEF

$\Rightarrow$ LAW OF SUPPLY $= \frac{\partial y_\ell(p)}{\partial p_\ell} \geq 0 \quad \forall \ell$.

SINGLE OUTPUT PRODUCTION TECHNOLOGY:

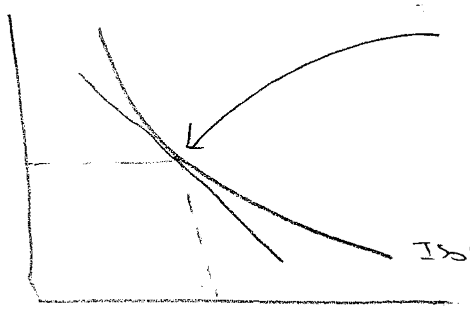
PRODUCTION FUNCTION $f: \mathbb{R}^n \to \mathbb{R}$

z = INPUT GOODS

w = INPUT PRICES $\in \mathbb{R}^n_{++}$

p = OUTPUT PRICE $\in \mathbb{R}_{++}$

ASSUME ISOQUANTS ARE CONVEX SO THAT f IS QUASI-CONCAVE.



$$MRTS_{\ell k} = \frac{\frac{\partial f(z)}{\partial z_\ell}}{\frac{\partial f(z)}{\partial z_k}} = -\text{ SLOPE OF ISOQUANT.}$$

<u>THM</u> (EX 5.B.3): FOR A SINGLE OUTPUT TECH, Y IS CONVEX IFF $f(\cdot)$ IS CONCAVE.

① <u>π-MP</u>: $\max_{z \in \mathbb{R}^n} \; pf(z) - w \cdot z$

<u>FOC</u>: $\mathcal{L} = pf(z) - w \cdot z$

$$\frac{\partial \mathcal{L}}{\partial z_\ell} = p\frac{\partial f(z)}{\partial z_\ell} - w_\ell \leq 0 \quad \forall \ell \quad \left(= 0 \text{ IF } z_\ell^* > 0 \right)$$

<u>SOC</u>: NOT NEEDED IF f IS CONCAVE.

$\Rightarrow$ AT INTERIOR SOLUTION $\dfrac{\frac{\partial f(z^*)}{\partial z_\ell}}{\frac{\partial f(z^*)}{\partial z_k}} = \dfrac{w_\ell}{w_k} \quad \forall \ell \neq k$

$\Rightarrow$ MRTS = PRICE RATIO.

SOLVE $\Rightarrow$ SOLUTION $z(p,w)$ = INPUT DEMAND

= FACTOR DEMAND

= UNCONDITIONAL INPUT DEMAND, ETC

VALUE FUNCTION $\pi(p) = p\,f(z(p,w)) - w \cdot z(p,w)$

= PROFIT FUNCTION.

(2) π-MP : $\max_{q \geq 0} \; pq - c(q)$

WHERE DOES COST FUNCTION $c(q)$

COME FROM ?

π-MAX $\Rightarrow$ FIRM WANTS TO MINIMIZE COST

= PRODUCE q AT MIN COST.

SOLVE :

COST MINIMIZATION PROBLEM

CMP : $\min_{z \in \mathbb{R}^L_+} \; w \cdot z$ s.t. $f(z) \geq q$

C.F. EMP.

FOC: $\mathcal{L} = w \cdot z + \mu\,[q - f(z)]$

(1) $\dfrac{\partial \mathcal{L}}{\partial z_\ell} = w_\ell - \mu \dfrac{\partial f(z)}{\partial z_\ell} \leq 0 \quad \forall \ell \quad [= 0 \text{ IF } z^*_\ell > 0]$

(2) $\dfrac{\partial \mathcal{L}}{\partial \mu} = q - f(z) = 0$

SOLUTION : $z(w,q)$ = CONDITIONAL INPUT (FACTOR) DEMAND

VALUE FUNCTION $c(w,q) = w \cdot z(w,q)$

= COST FUNCTION

THM: SUPPOSE Y IS A PRODUCTION SET FOR A SINGLE OUTPUT TECHNOLOGY & IS CLOSED & SATISFIES FREE DISPOSAL PROP THEN

① $z(\cdot)$ IS HD0 IN w

② IF "ISOQUANTS ARE CONVEX" THEN $z(\cdot)$ IS CONVEX. IF S. CONVEX THEN $z(\cdot)$ IS A SINGLETON

③ $c(\cdot)$ IS HD1 IN w & NON-DECREASING IN q

④ $c(\cdot)$ IS CONCAVE IN w.

⑤ IF $z(\cdot)$ IS A SINGLETON, THEN

$$z_\ell(w,q) = \frac{\partial c(w,q)}{\partial w_\ell} \quad \forall \ell \quad \text{(SHEPARD'S LEMMA)}$$

&

$$z_\ell(p,w) = -\frac{\partial \pi(p,w)}{\partial w_\ell} \quad \forall \ell \quad \text{(HOTELLING'S LEMMA)}$$

⑥ $D_w z(w,q)$ IS SYMMETRIC, N.S.D & SATISFIES $D_w z(w,q)\, w = 0$

⑦ IF $f(\cdot)$ IS HD1 (CRS) THEN $c(\cdot)$ & $z(\cdot)$ ARE HD1.

⑧ IF $f(\cdot)$ IS CONCAVE THEN $c(\cdot)$ IS A CONVEX FUNC. OF $\underline{\underline{q}}$

REMARK: ⑧ $\Rightarrow$ MC IS NON-DECREASING IN q.

THEORY OF THE FIRM:

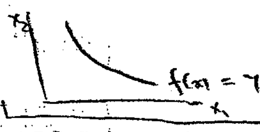
- Q: What is a firm?

For our purposes, firm is just a black box that takes inputs and transforms them into outputs.

⇒ ALL WE CARE ABOUT IS THE FIRM'S PRODUCTION FUNCTION $f: \mathbb{R}^n_+ \to \mathbb{R}_+$.

IF FIRM USES $x=(x_1,\dots,x_n)$ units of inputs then firm produces $f(x)$ unit of output.

- ASSUME $f(\cdot)$ IS QUASI-CONCAVE (THAT ISOQUANT IS CONVEX)



- Q: What do firms do?

FIRMS MAXIMIZE PROFIT

$$\max_y \; py - c(y), \quad \text{WHERE } c \text{ IS A COST FUNCTION.}$$

FOC LEADS TO FAMILIAR

$$p = c'(y)$$
$$\Leftrightarrow MR = MC.$$

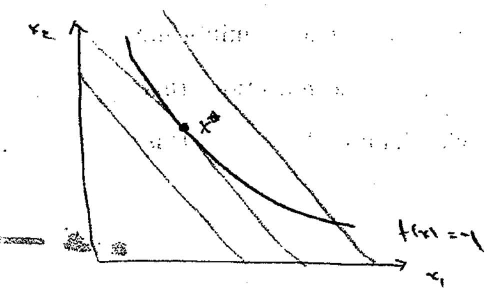
- WE'LL EXAMINE THIS IN DETAIL.

FIRST, WHERE DOES COST FUNCTION COME FROM?

CONSIDER FIRM'S INPUT DECISION:

FIRMS TAKE INPUT PRICES $w = (w_1,\dots,w_n)$ AS GIVEN AND TRIES TO PRODUCE SOME OUTPUT y AT MINIMUM COST

IE. $\min_x \; w\cdot x$ s.t. $f(x)=y$



NOTE: THIS IS EXACTLY THE SAME PROBLEM AS EXPENDITURE FUNCTION.

⇒ SOLVE IN THE SAME WAY

$\Rightarrow$ SOLVE THE SAME WAY.

F.O.C : $\dfrac{\frac{\partial f(x)}{\partial x_i}}{\frac{\partial f(x)}{\partial x_j}} = \dfrac{w_i}{w_j}$ MRTS = PRICE RATIO

+ OUTPUT CONSTRAINT.

$\Rightarrow$ GET SOLUTION $x(w,y)$

THIS IS CALLED CONDITIONAL INPUT DEMANDS.

- NOW SUBSTITUTE INTO OBJECTIVE FUNCTION

$\Rightarrow$ COST FUNCTION. < VALUE FUNCTION >

$c(w,y) = w \cdot x(w,y)$

NOTE : WHEN w IS FIXED AND WE DON'T NEED TO EMPHASIZE THAT COST DEPENDS ON w, WRITE $c(y)$.

- PROPERTIES OF COST FUNCTION :

① $c(w,y)$ IS CONCAVE IN w.

② $\dfrac{\partial c(w,y)}{\partial w_i} = x_i^*(w,y) \;\forall i$ < SAME AS SHEPARD'S LEMMA >

+ MORE

$\rightarrow$ DO EXAMPLE: Δ

- BACK TO PROFIT MAXIMIZATION :

$$\max_x \; p f(x) - w \cdot x$$

$$\Leftrightarrow \max_y \; py - c(w,y)$$

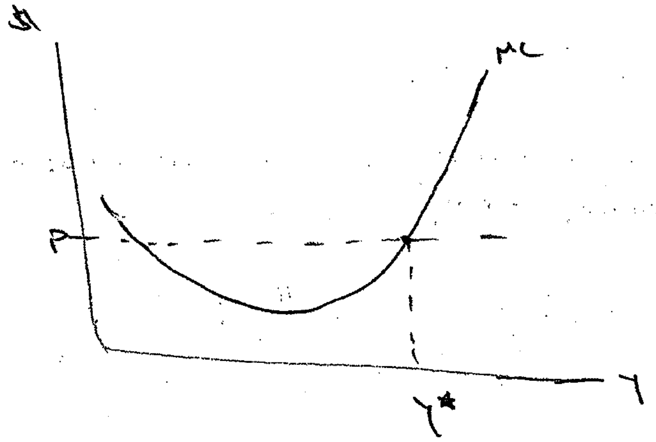
F.O.C : $p - \dfrac{\partial c(w,y)}{\partial y} = 0 \Leftrightarrow p = \dfrac{\partial c(w,y)}{\partial y}$

S.O.C : $-\dfrac{\partial^2 c(w,y)}{\partial y^2} \leq 0 \Leftrightarrow \dfrac{\partial^2 c(w,y)}{\partial y^2} \geq 0$.

So, the optimal output decision is y^*.

WHERE $\frac{\partial c(w,y^*)}{\partial y} = p$ MR = MC

& $\frac{\partial}{\partial y}\left[\frac{\partial c(w,y^*)}{\partial y}\right] \geq 0$ MC IS INCREASING.



- SOLUTION TO π-MAX PROBLEM

$\max_{y} \; p \cdot y - c(w,y) \Rightarrow y^*(p,w)$ SUPPLY FUNCTION

$\Leftrightarrow \max_{x} \; p \cdot f(x) - w \cdot x \Rightarrow x^*(p,w)$ INPUT DEMAND FUNCTION

NOTE: $y(p,w) = f(x(p,w))$ $x(p,w) = x(w, y(p,w))$

NOTE $x(p,w)$ IS THE (UNCONDITIONAL) INPUT DEMAND FUNCTION
$x(w,y)$ IS THE CONDITIONAL INPUT DEMAND FUNCTION

$\pi(p,w) = p \cdot f(x^*(p,w)) - p \cdot x^*(p,w)$ IS THE PROFIT FUNCTION

- PROPERTIES OF PROFIT FUNCTION

HOTELLING'S LEMMA:

$\frac{\partial \pi(p,w)}{\partial p} = y(p,w)$ $-\frac{\partial \pi(p,w)}{\partial w_i} = x_i(p,w) \;\; \forall i.$

TUES LECTURE

EXAMPLE: $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$

- To derive conditional input demand & cost function given input prices w & output y

$$\min_x \; w \cdot x \quad \text{s.t.} \quad f(x) = y \quad \Leftrightarrow \quad \max_x \; -w_1 x_1 - w_2 x_2 + \lambda [x_1^{1/3} x_2^{1/3} - y]$$

$$\mathcal{L} = -w_1 x_1 - w_2 x_2 + \lambda [x_1^{1/3} x_2^{1/3} - y] \qquad \mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda [y - x_1^{1/3} x_2^{1/3}]$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= -w_1 + \lambda \tfrac{1}{3} x_1^{-2/3} x_2^{1/3} = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= -w_2 + \lambda \tfrac{1}{3} x_1^{1/3} x_2^{-2/3} = 0 \end{aligned}\right] \Rightarrow \frac{x_2}{x_1} = \frac{w_1}{w_2}$$

$$\Rightarrow x_2 = \frac{w_1}{w_2} x_1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x_1^{1/3} x_2^{1/3} - y = 0$$

$$\Rightarrow x_1^{1/3} \left(\frac{w_1}{w_2} x_1\right)^{1/3} = y$$

$$x_1^{2/3} = \left(\frac{w_2}{w_1}\right)^{1/3} y$$

$$x_1(w, y) = \left(\frac{w_2}{w_1}\right)^{1/2} y^{3/2}$$

$$x_2 = \left(\frac{w_1}{w_2}\right)\left(\frac{w_2}{w_1}\right)^{1/2} y^{3/2}$$

$$x_2(w, y) = \left(\frac{w_1}{w_2}\right)^{1/2} y^{3/2}$$

Cost function
$$c(w, y) = w_1 \left(\frac{w_2}{w_1}\right)^{1/2} y^{3/2} + w_2 \left(\frac{w_1}{w_2}\right)^{1/2} y^{3/2}$$
$$= \left[(w_1 w_2)^{1/2} + (w_1 w_2)^{1/2}\right] y^{3/2}$$
$$= 2 (w_1 w_2)^{1/2} y^{3/2}$$

VERIFY SHEPARD'S LEMMA: $\dfrac{\partial c(w, y)}{\partial w_i} = x_i(w, y)$

$$\frac{\partial c(w, y)}{\partial w_1} = 2\left(\tfrac{1}{2}\right)(w_1 w_2)^{-\frac{1}{2}} w_2 y^{3/2}$$
$$= \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}} y^{3/2} = x_1(w, y) \quad \checkmark$$

CHECK $\dfrac{\partial c(w, y)}{\partial w_2} = x_2(w, y)$ also.

- FIND UNCONDITIONAL INPUT DEMAND, SUPPLY FUNCTION, PROFIT FUNCTION:

(B) CAN SOLVE $\max_x \; pf(x) - w \cdot x$ DIRECTLY

OR (A) $\max_y \; py - c(w,y)$

LETS DO (A): $\max_y \; py - 2(w_1 w_2)^{1/2} y^{3/2}$

FOC

$$p - 2(w_1 w_2)^{1/2} \frac{3}{2} y^{1/2} = 0$$

$$3(w_1 w_2)^{1/2} y^{1/2} = p$$

$$y = \left[\frac{p}{3(w_1 w_2)^{1/2}}\right]^2 = \frac{p^2}{9 w_1 w_2}$$

So, SUPPLY FUNCTION IS $y(p,w) = \dfrac{p^2}{9 w_1 w_2}$

- TO FIND UNCONDITIONAL INPUT DEMAND

LOOK FOR $x(p,w) = x(w, y(p,w))$

$$\Rightarrow \underline{x_1(p,w)} = \left(\frac{w_2}{w_1}\right)^{1/2} \left(\frac{p^2}{9 w_1 w_2}\right)^{3/2}$$

$= \left(\frac{w_2}{w_1}\right)^{1/2} [y(p,w)]^{3/2}$ //

$$= \frac{(w_2)^{1/2} (p^2)^{3/2}}{(w_1)^{1/2} (9)^{3/2} (w_1)^{3/2} (w_2)^{3/2}}$$

$$= \frac{p^3}{27 w_1^2 w_2}$$

$$\underline{x_2(p,w)} = \left(\frac{w_1}{w_2}\right)^{1/2} \left(\frac{p^2}{9 w_1 w_2}\right)^{3/2} = \frac{p^3}{27 w_2^2 w_1}$$

PROFIT FUNCTION IS: $\pi(p,w) = p \cdot y(p,w) - w_1 x_1(p,w) - w_2 x_2(p,w)$

$$= p\left(\frac{p^2}{9 w_1 w_2}\right) - w_1\left(\frac{p^3}{27 w_1^2 w_2}\right) - w_2\left(\frac{p^3}{27 w_2^2 w_1}\right)$$

$$= \frac{p^3}{9 w_1 w_2} - \frac{p^3}{27 w_1 w_2} - \frac{p^3}{27 w_2 w_1}$$

$$= \frac{p^3}{27 w_1 w_2} \; ..$$

VERIFY HOTELLING'S LEMMA:

① $\frac{\partial \pi}{\partial p} = y(p,w)$

② $-\frac{\partial \pi}{\partial w_1} = x_1(p,w)$

③ $-\frac{\partial \pi}{\partial w_2} = x_2(p,w)$

Check ①: $\pi(p,w) = \frac{p^3}{27 w_1 w_2}$

① $\frac{\partial \pi}{\partial p} = \frac{3p^2}{27 w_1 w_2} = \frac{p^2}{9 w_1 w_2} = y(p,w)$ ✓

② $-\frac{\partial \pi}{\partial w_1} = -\frac{\partial}{\partial w_1}\left[\left(\frac{p^3}{27 w_2}\right) w_1^{-1}\right]$

$= -\left(\frac{p^3}{27 w_2}\right)\left(-w_1^{-2}\right)$

$= \frac{p^3}{27 w_1^2 w_2} = x_1(p,w)$ ✓

③ Check $-\frac{\partial \pi}{\partial w_2} = x_2(p,w)$ ALSO. ✓

LET'S DO APPROACH (B) & SHOW THAT SOLN ARE SAME.

$\max_{x}\ p f(x) - w \cdot x$

$\iff \max_{x_1, x_2}\ p\, x_1^{1/3} x_2^{1/3} - w_1 x_1 - w_2 x_2$

FOC: ① $\frac{\partial [\ ]}{\partial x_1} = p\left(\frac{1}{3}\right) x_1^{-2/3} x_2^{1/3} - w_1 = 0$

② $\frac{\partial [\ ]}{\partial x_2} = p\left(x_1^{1/3}\right)\left(\frac{1}{3}\right) x_2^{-2/3} - w_2 = 0$

DIVIDE $\Rightarrow \frac{p\left(\frac{1}{3}\right) x_1^{-2/3} x_2^{1/3}}{p\left(x_1^{1/3}\right)\left(\frac{1}{3}\right) x_2^{-2/3}} = \frac{w_1}{w_2}$

$\Rightarrow \frac{x_2}{x_1} = \frac{w_1}{w_2} \Rightarrow x_2 = \frac{w_1}{w_2} x_1$ (*)

Substitute (*) into (1) gives

$$p\left(\tfrac{1}{3}\right) x_1^{-2/3}\left(\frac{w_1}{w_2}x_1\right)^{1/3} - w_1 = 0$$

$$p\left(\tfrac{1}{3}\right)\left(\frac{w_1}{w_2}\right)^{1/3} x_1^{-1/3} = w_1$$

$$x_1^{1/3} = \frac{p\left(\frac{1}{3}\right)\left(\frac{w_1}{w_2}\right)^{1/3}}{w_1}$$

$$x_1 = \frac{p^3\left(\frac{1}{27}\right)\frac{w_1}{w_2}}{w_1^3}$$

$$\Rightarrow x_1(p,w) = \frac{p^3}{27 w_1^2 w_2} \quad \checkmark$$

$$\Rightarrow x_2(p,w) = \frac{w_1}{w_2}\left(\frac{p^3}{27 w_1^2 w_2}\right) = \frac{p^3}{27 w_1 w_2^2} \quad \checkmark$$

$\Rightarrow$ Supply function is

$$y(p,w) = [x_1(p,w)]^{1/3}\,[x_2(p,w)]^{1/3}$$

$$= \left(\frac{p^3}{27 w_1^2 w_2}\right)^{1/3}\left(\frac{p^3}{27 w_1 w_2^2}\right)^{1/3}$$

$$= \frac{p}{3 w_1^{2/3} w_2^{1/3}} \cdot \frac{p}{3 w_1^{1/3} w_2^{2/3}}$$

$$= \frac{p^2}{9 w_1 w_2} \quad \checkmark$$

So, (A) & (B) gives same answer.

REMINDER: EXAM TUES

COVERS EVERYTHING TO HERE.

TEXT BOOK ONLY SO FAR AS IT OVERLAPS WITH LECTURES

EXAMS : [redacted]