Notes on General Equilibrium

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CAUTION: This a first draft of lectures notes on general equilibrium theory for graduate students. They cover materials from Chapters 15-17 of Mas-Colell, Whinston, and Green (1995). Being our first draft, these notes are incomplete and likely contain more than the usual amount of typos and errors. Please read with caution and feel welcome to alert us if you find any mistakes.

1 Introduction

When we study competitive markets in intermediate microeconomics, we typically examine the market for one good at a time. This simplification helps us develop a basic understanding of how markets work and the role price plays in mediating the desires of the (price-taking) sellers and buyers. In particular, the price adjusts to equalize the demand of the buyers and the supply of the sellers. When the demand and the supply are exactly balanced, we call that a competitive equilibrium. We also observe that a competitive equilibrium is "efficient" in that it maximizes social welfare.

However, despite its usefulness, studying a single market in isolation is limiting in one very important way. A partial equilibrium analysis like this ignores the cross effect one market has on other markets and the feedback effect that could in turn have in the original market. As an example, consider the market for coffee. Suppose initially there is an excess supply of coffee in the coffee market while the market for tea is in equilibrium. If we are only interested in a partial equilibrium analysis, we would say that the price of coffee will fall to equilibrate the coffee market, and that will be the end of the story. However, suppose we are interested in studying the coffee and tea markets together. Then since coffee and tea are substitute goods, a fall in coffee price will shift the demand for tea to the left, creating an excess supply of tea. If the price of tea falls to bring the tea market back into an equilibrium, then now the demand for coffee will shift to the left and create an excess supply of coffee again. Thus, it is not obvious that there are prices of tea and coffee that simultaneously equilibriate the two markets. Moreover, The efficiency properties of such equilibrium, if it exists, is even less clear. The theory of general equilibrium seeks to answer these questions by studying interrelated markets together, taking into account their cross effects and feedback effects.

While general equilibrium analysis raises a host of deep and interesting theoretical questions, it also has important practical relevance. As the following tax example illustrates, a partial equilibrium analysis can sometimes lead us seriously astray.

Example 1.1 (Tax incidence (MWG 15.E))**.** Consider a national economy with N many cities (assume N is large). There are two goods in the economy, leisure/labor and a consumption good. The price of the consumption good is normalized to 1 ($p = 1$), and w denotes the price of labor (wage). Each city has one firm that uses labor to produce the consumption good, which is sold in a common national market. The firms are identical, with production function $f(z)$ that is increasing and strictly concave. Individuals in the economy collectively have M units of time in total, which can be used in leisure or labor. To keep things simple, assume that individuals do not care about leisure and only derive utility from the consumption good. This implies that they always supply M units of labor in total and use the income to buy the consumption good. Assume that the firms cannot relocate but the individuals are allowed to move freely. Finally assume that, aside from the wage offered by the firm in the city, the cities are identical so that an individual will work for whichever firm offers the highest wage.

Suppose the city economies are all in equilibrium initially. Firm j's profit maximization problem is

$$\max_{z_j} \; pf(z_j) - w_j z_j,$$

where w_j is the wage offered by firm j and z_j is the firm's labor usage. The firm's labor demand is given by the first order condition (we have substituted in $p = 1$),

$$f'(z_j) = w_j.$$

Since cities are identical and there is free mobility, all the firms must offer the same wage in equilibrium. (Otherwise, the firm offering the highest wage will attract all the workers). Thus, initially,

$$w_1^* = w_2^* = w_3^* = \cdots = w_N^* = \bar{w}.$$

Because the firms are identical and the wages are the same, the every firm employs the same amount of labor.

$$z_1^* = z_2^* = z_3^* = \cdots = z_N^* = \frac{M}{N}.$$

Thus, the equilibrium wage and each firm's profit are given by

$$w_j^* = \bar{w} = f'\left(\tfrac{M}{N}\right) \quad \text{and} \quad \pi_j(w_j^*) = \pi(\bar{w}) = f\left(\tfrac{M}{N}\right) - f'\left(\tfrac{M}{N}\right)\left(\frac{M}{N}\right).$$

Now, suppose city 1 needs to raise revenue and is considering instituting an income tax (per unit tax in the labor market) in its jurisdiction. How will the tax burden be shared by the workers and the firms?

Partial equilibrium analysis: Suppose we assume that what happens in city 1 does not affect other cities because the size of one city's labor market is small relative to the national economy (N is large). So we look at the labor market in the city 1 in isolation. Since the effect of tax is independent of whether it's being collected from the demand or the supply side, let's assume that it is collected from the firm. Let w_1 be the wage received by the workers and $w_1 + t$ the wage (plus tax) paid by firm 1. The firm's labor demand now solves

$$f_1'(z_1) = w_1 + t.$$

Since labor is mobile and the wages in the other cities are still $\bar{w}$, the labor supply is effectively perfectly elastic to the firm in city 1:

$$L^s(w_1) = \begin{cases} 0 & \text{if } w_1 < \bar{w} \\ [0, M] & \text{if } w_1 = \bar{w} \\ M & \text{if } w_1 > \bar{w}. \end{cases}$$

Thus, the equilibrium (after-tax) wage in city 1 is once again $w_1 = \bar{w}$, and the amount of labor employed in city 1 is found by solving.

$$f_1'(\tilde{z}_1) = \bar{w} + t,$$

as illustrated in figure 1.1 (in the figure, labor demand is drawn as a line for convenience.) Since labor employed in city 1 is $\tilde{z}_1$, the remaining labor, $z_1^* - \tilde{z}_1$, will move to other cities. Because we are assuming that wages in other cities are unaffected by the tax in city 1, relocating workers would presumably receive wage $\bar{w}$ and be equally well off as before. Thus, in this partial equilibrium analysis, we conclude that the entire burden of the tax is borne by the firm in city 1. In particular, the tax has no negative effect on the workers.

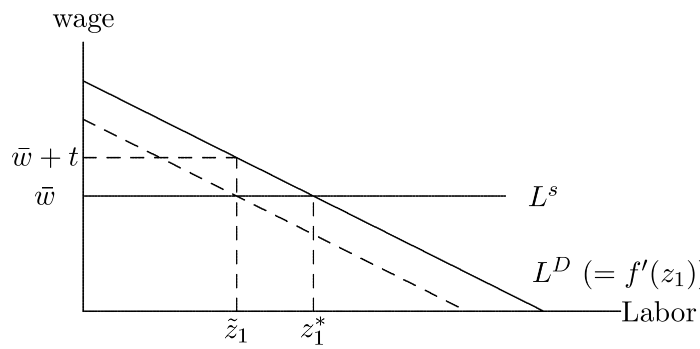

Figure 1.1: Partial equilibrium analysis.

General equilibrium analysis: Now, suppose we take a more comprehensive view and incorporate the labor market in all the cities into the analysis. Free mobility means that after-tax wages must be the same across all the cities. We denote the equilibrium after-tax wage as $w(t)$. Since firms 2,...,N are identical they must be using the same amount of labor in equilibrium. Thus, the equilibrium conditions for the entire economy can be stated as:

$$f'(z_1(t)) = w(t) + t \tag{1}$$
$$f'(z_2(t)) = w(t) \tag{2}$$
$$z_1(t) + (N-1)z_2(t) = M. \tag{3}$$

Condition (1) states that firm 1 is using its profit maximizing amount of labor. Condition (2) states that firm 2 (and hence every firm $j \neq 1$) is also choosing its profit maximizing amount of labor. Condition (3) sates that the labor market clears at the national level.

The effect of imposing a small amount of tax, when there was no tax before, on the equilibrium wage is $\frac{dw(t)}{dt}\big|_{t=0}$, which can be obtained in the following way. First, differentiate condition (2) with respect to t and then evaluate it at $t = 0$.

$$\frac{d}{dt}\Big[f'(z_2(t))\Big]_{t=0} = \frac{d}{dt}\Big[w(t)\Big]_{t=0}$$
$$\Big[f''\big(z_2(t)\big)\big(z_2'(t)\big)\Big]_{t=0} = \Big[w'(t)\Big]_{t=0}$$
$$f''\big(z_2(0)\big)\big(z_2'(0)\big) = w'(0)$$
$$f''\big(\tfrac{M}{N}\big)\big(z_2'(0)\big) = w'(0) \quad \text{since } z_2(0) = \tfrac{M}{N}. \tag{4}$$

Next, substitute condition (3) into condition (1). Then differentiate the resulting identity with respect to t and evaluate it at $t = 0$ to obtain

$$\begin{aligned}
f'\big(M-(N-1)z_2(t)\big) &= w(t)+t \\
\Longrightarrow \quad \frac{d}{dt}\Big[f'\big(M-(N-1)z_2(t)\big)\Big]_{t=0} &= \frac{d}{dt}\Big[w(t)+t\Big]_{t=0} \\
\Big[f''\big(M-(N-1)z_2(t)\big)\big(-(N-1)z_2'(t)\big)\Big]_{t=0} &= \Big[w'(t)+1\Big]_{t=0} \\
-f''\big(M-(N-1)z_2(0)\big)\big((N-1)z_2'(0)\big) &= w'(0)+1 \\
-f''\big(M-(N-1)\tfrac{M}{N}\big)\big((N-1)z_2'(0)\big) &= w'(0)+1 \\
-f''\big(\tfrac{M}{N}\big)\big((N-1)z_2'(0)\big) &= w'(0)+1 \\
-(N-1)w'(0) &= w'(0)+1 \quad \text{by equation (4)} \\
w'(0) &= -\frac{1}{N} < 0. \qquad (5)
\end{aligned}$$

Thus, the general equilibrium analysis shows that the earlier partial equilibrium analysis was critically flawed and that all the workers in the economy are negatively affected. Worse yet, the following shows that the workers in fact bear all the burdens of the tax. The aggregate profit of the firms in the economy is

$$\text{aggregate profit}(t) = \pi\big(w(t)+t\big) + (N-1)\pi\big(w(t)\big).$$

As before, differentiate this with respect to t and then evaluate it at $t = 0$ to obtain the marginal effect of the tax:

$$\begin{aligned}
\frac{d}{dt}\Big[\text{agg. profit}(t)\Big]_{t=0} &= \Big[\pi'\big(w(t)+t\big)\big(w'(t)+1\big) + (N-1)\pi'\big(w(t)\big)w'(t)\Big]_{t=0} \\
&= \pi'\big(w(0)\big)\big(w'(0)+1\big) + (N-1)\pi'\big(w(0)\big)w'(0) \\
&= \pi'\big(w(0)\big)\Big(w'(0)+1+(N-1)w'(0)\Big) \\
&= \pi'\big(\bar{w}\big)\Big(Nw'(0)+1\Big) \quad \text{since } w(0)=\bar{w} \\
&= \pi'\big(\bar{w}\big)\Big(N\left(-\tfrac{1}{N}\right)+1\Big) \quad \text{by equation (5)} \\
&= 0.
\end{aligned}$$

Therefore, the tax has no effect on the firms' aggregate profit, which implies that it is the workers that bear the full brunt of the tax. □

2 Arrow Debreu Economy

The ambitious aim of the general equilibrium theory makes it a difficult subject. To keep it manageable, we will start with the formulation of the general model but give only a brief description of the central results. We will then study these results in the context of three specific models in Section 3. Once we have gain some experience with these simpler general equilibrium models, we will return to the central results in greater detail, as well as study some important ancillary results.

2.1 Specification of the economy

The general model is an economy consisting of L goods, I consumers and J firms. Although we will make further restrictions as we go along, here are the basic set up and notations.

- L goods, denoted $\ell = 1, 2, .., L$.
- I individuals, denoted $i = 1, 2, ..., I$.

  Individual i's consumption set, which is the set of all possible bundles she can consume, is $X_i \subset \mathbb{R}^L$. Individual i's consumption bundle is denoted $x_i = (x_{1i}, x_{2i}, ..., x_{Li})$, where $x_{\ell i}$ is the amount of good ℓ she has.

  Each individual i has a preference ordering $\succsim_i$ over X_i. The corresponding utility function is denoted $u_i(\cdot)$.

  Each individual i is endowed with an initial resource (called endowment), $\omega_i = (\omega_{1i}, \omega_{2i}, ..., \omega_{Li}) \in \mathbb{R}^L$. The aggregate endowment, or the total resource, of the economy is $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, ..., \bar{\omega}_L) = \sum_{i=1}^{I} \omega_i$. Note that ω_i is individual i's endowment bundle while $\bar{\omega}_\ell$ is the aggregate endowment of good ℓ in the economy. Also, note that ω being used here is the Greek letter "omega," not the Roman alphabet "double u" (w), which will be used later for either "wage" or "wealth."

- J firms, denoted $j = 1, .., J$. Firm j's production set, which is the set of all possible production plans for the firm, is denoted $Y_j \subset \mathbb{R}^L$. Firm j's production plan is $y_j = (y_{1j}, ..., y_{Lj})$, where the magnitude of $y_{\ell j}$ is the amount of good ℓ that is used as input or produced as output. Negative $y_{\ell j}$ denotes input while positive $y_{\ell j}$ denotes output.
- Firms are owned by the individuals. Individual i's ownership share of firm j is $\theta_{ij} \geq 0$. For all firm j, $\sum_{i=1}^{I} \theta_{ij} = 1$.

Thus, at an abstract level, an Arrow Debreu economy is a specification:

$$\Big(X_i,\ \succsim_i, \text{ and } \omega_i \text{ for all } i;\ Y_j \text{ for all } j; \text{ and } \theta_{ij} \text{ for all } i \text{ and } j \Big).$$

An *allocation* is a vector that specifies consumption plans for all the individuals and production plans for all the firms. That is, $(x, y) = (x_1, ..., x_I, y_1, ..., y_J)$, where $x_i \in X_i$ for all i and $y_j \in Y_j$ for all j. An allocation in an Arrow Debreu economy is said to be *feasible* if

$$\sum_{i=1}^{I} x_i = \bar{\omega} + \sum_{j=1}^{J} y_j.$$

We can roughly think of an allocation as the "tangible" state of the economy (the intangible part being prices). We want to be able to predict what state the economy will be in. Clearly, the economy can only be in a feasible allocation. To be able to say more, we need to make further assumptions on the model.

When we studied consumer theory in intermediate microeconomics, we ensured that the preference orderings are well behaved by assuming that certain properties are satisfied. A typical set of properties that were assumed is:

1. completeness: For any x_i and x'_i, we have $x_i \succsim_i x'_i$, $x'_i \succsim_i x_i$, or both.
2. transitivity: If $x_i \succsim_i x'_i$ and $x'_i \succsim_i x''_i$, we have $x_i \succsim_i x''_i$.
3. continuity: For every sequence of bundles $x_i^n \to x_i$ and $\hat{x}_i^n \to \hat{x}_i$, with $x_i^n \succsim \hat{x}_i^n$ for all n, we have $x_i \succsim_i \hat{x}_i$.
4. monotonicity: If $x_{\ell i} > x'_{\ell i}$ for all $\ell = 1, ..., L$, then $(x_{1i}, ..., x_{Li}) \succ (x'_{1i}, ..., x'_{Li})$.
5. convexity: If $x \succsim y$ then $\alpha x + (1 - \alpha)y \succsim y$ for all $0 \leq \alpha \leq 1$.

A preferences ordering satisfying completeness and transitivity is called *rational*. Continuity requires that the preference ordering is preserved under the limit operation and roughly means that preference ordering does not make a sudden jump. As we have learned in consumer theory, these three properties together imply that there is a (continuous) utility function representing the preference ordering.

Monotonicity property is a "desirability" property. There are two other commonly used desirability properties. Since consumption bundles are vectors, it will be convenient to introduce the following notation. Given two N-dimensional vectors $a = (a_1, ..., a_N)$ and $b = (b_1, ..., b_N)$, we write $a \gg b$ to mean $a_n > b_n$ for all $n = 1, ..., N$ and write $a > b$ to mean $a_n \geq b_n$ for all n and $a_n > b_n$ for some n.

Definition 2.1. A preference ordering $\succsim_i$ is said to be

1. *strongly monotone* if $x'_i > x_i \implies x'_i \succ_i x_i$.
2. *(weakly) monotone* if $x'_i \gg x_i \implies x'_i \succ_i x_i$.
3. *locally non-satiated* if for every $x_i \in X_i$ and $\varepsilon > 0$, there is $x'_i \in X_i$ such that $\|x'_i - x_i\| < \varepsilon$ and $x'_i \succ_i x_i$.

□

Strong monotonicity requires that the individual is happier if she is given more of any good, while (weak) monotonicity requires that the individual is happier if she is given more of everything. Local non-satiation requires that no matter what the consumer has currently, there is a nearby bundle that is better. This is a desirability assumption that applies to "bads" like pollution since it does not require having more to be better. For example, figure 2.1 depicts a locally non-satiated preference in which utility increases in the direction of the arrow, making good 1 a "bad" and good 2 a "good" for individual i.

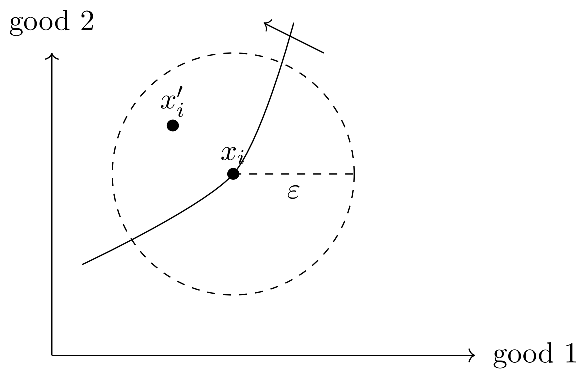

Figure 2.1: Local non-satiation: $x_i' \succ x_i$.

Exercise: Let the consumption set be $\mathbb{R}^L_+ = \{(x_1, x_2, ..., x_L) : x_\ell \geq 0 \text{ for all } \ell\}$. Show that a preference that is strongly monotone is weakly monotone, and a preference that is weakly monotone is locally non-satiated.

□

Convexity property requires that a mixture of two bundles is at least as good as the worse of the two and captures the taste for diversification. Like monotonicity, convexity also has a "weak" and a "strong" version. There are multiple, equivalent ways of defining them, and we will give the definition using the upper contour set, which is the area above an indifference curve, including the curve itself.[1] Formally, an *upper contour set* of $\succsim_i$ corresponding to x_i is $\{x_i' \in X_i : x_i' \succsim_i x_i\}$.

Definition 2.2. A preference ordering $\succsim_i$ is said to be

1. *(weakly) convex* if the upper contour set corresponding to x_i is a convex set for every x_i. That is, $x_i' \succsim_i x_i$ and $x_i'' \succsim_i x_i \implies \alpha x_i' + (1-\alpha)x_i'' \succsim_i x_i$ for all $\alpha \in [0,1]$.

2. *strongly convex* if the upper contour set corresponding to x_i is a strictly convex set for every x_i. That is, $x_i' \succsim_i x_i$ and $x_i'' \succsim_i x_i \implies \alpha x_i' + (1-\alpha)x_i'' \succ_i x_i$ for all $\alpha \in (0,1)$.

□

We will maintain completeness, transitivity, and continuity assumptions through out these lecture notes (unless stated otherwise) and assume some version of desir-

[1]In the following, $[a, b]$ is the closed interval between a and b, including the end points a and b. In contrast, (a, b) is the open interval without the end points.

ability and convexity as needed. In particular, assuming the first three properties allows us to use the preference ordering and the utility function that represents it interchangeably. Assumptions on the production technologies are somewhat less interesting and will be made as they are needed.

2.2 Behavioral assumptions

Let $p = (p_1, ..., p_L) \in \mathbb{R}^L$ be a price vector (for now we allow prices to be negative). Since a production plan is a *netput* vector, where negative components have the interpretation of being inputs and positive components outputs, the profit derived from production plan y_j is $p \bullet y_j$. For example, production plan $y_j = (3, -5, 0, -7, 6)$ means the firm will use 5 units of good 2 and 7 unites of good 4 to produce 3 units of good 1 and 6 units of good 5. The profit derived from this plan when prices are $p = (p_1, ..., p_5) \gg 0$ is

$$p \bullet y_j = (p_1, p_2, p_3, p_4, p_5) \bullet (3, -5, 0, -7, 6) = \underbrace{(3p_1 + 6p_5)}_{\text{revenue}} - \underbrace{(5p_2 + 7p_4)}_{\text{cost}}.$$

The firms in the economy is assumed to maximize profit, taking prices as given. Thus, each firm j solves

$$\max_{y_j \in Y_j} \; p \bullet y_j.$$

Let $y_j(p)$ be the solution (assuming one exists) to the profit maximization problem. This is called firm j's *supply correspondence* and is technically a set since there could be more than one profit maximizing plan. Let $\pi_j(p) = p \bullet y_j$, where $y_j \in y_j(p)$, be the firm's *profit function*.

Firms remit profits to their owners. Given any price vector p and any profile of production plans $y = (y_1, ..., y_J)$ (not necessarily the profit maximizing plans), individual i's budget set is

$$B_i(p, y) = \left\{ x_i \in X_i : \; p \bullet x_i \leq p \bullet \omega_i + \sum_{j=1}^{J} \theta_{ij} p \bullet y_j \right\}.$$

Individuals are assumed to "maximize preference," taking prices as given. Thus, individual i solves

$$\text{choose } x_i^* \in B_i(p, y) \text{ such that } x_i^* \succsim_i x_i \text{ for all } x_i \in B_i(p, y).$$

Or, equivalently,

$$\max_{x_i \in B_i(p,y)} \; u_i(x_i).$$

From this point on, we will use the phrases "preference maximizing" and "utility maximizing" interchangeably. Let $x_i(p, y)$ be the solution (assuming one exists) to the utility maximization problem. This is called individual i's *(Marshallian) demand correspondence* and is technically a set since there could be more than one utility maximizing bundle.

In this economy, only the relative prices, rather than their levels, matter in the sense that scaling a price vector by a positive constant does not change the firms' profit maximizing behavior or the individuals' utility maximizing behavior.

Theorem 2.3. *In an Arrow-Debreu economy, the firms' supply correspondences and the individuals' demand correspondences are homogeneous of degree zero in prices. That is, for all* $\alpha > 0$,

$$y_j(\alpha p) = y_j(p) \quad \text{and} \quad x_i(\alpha p, y) = x_i(p, y).$$

Exercise: Verify Theorem 2.3.

□

An equilibrium in the economy occurs when all the individuals' utility maximizing consumption plans and the firms' profit maximizing production plans are compatible with one another.

Definition 2.4. A *Walrasian (or, competitive) equilibrium* in an Arrow Debreu economy is a price vector p^* and an allocation (x^*, y^*) such that

1. For all firm j, $y_j^* \in y_j(p^*)$, (profit maximization)
2. For all individual i, $x_i^* \in x_i(p^*, y^*)$, and (preference maximization)
3. $\sum_{i=1}^{I} x_i^* = \bar{\omega} + \sum_{j=1}^{J} y_j^*$. (market clearance)

□

The following subsection gives a brief discussion of the central results concerning the existence of Walrasian equilibrium and its welfare properties.

2.3 Brief discussion of the central results

Theorem 2.6 gives the conditions under which a Walrasian equilibrium is guaranteed to exist. It shows that economies satisfying an arguably reasonable set of conditions has a steady state. We will defer its proof, which uses the mathematical result known as fixed point theorem, to later after discussing the role of the stated conditions in the context of specific models in Section 3. One of the conditions, free disposal, means that it is possible for a firm to discard goods.

Definition 2.5. A production set Y_j satisfies *free disposal* if

$$y_j \in Y_j \text{ and } y_j' \leq y_j \implies y_j' \in Y_j.$$

□

Theorem 2.6. *Suppose an Arrow-Debreu economy satisfies the following conditions.*

1. *For all i, $X_i \subset \mathbb{R}^L$ is closed and convex, and $\omega_i \gg \hat{x}_i$ for some $\hat{x}_i \in X_i$.*
2. *For all i, $\succsim_i$ is complete, transitive, continuous, locally non-satiated, and convex.*
3. *For all j, Y_j is closed, convex, includes the origin, and satisfies free-disposal.*
4. *The set of feasible allocation is compact (closed and bounded).*

Then a Walrasian equilibrium exists.

The condition $\omega_i \gg \hat{x}_i$ for some $\hat{x}_i \in X_i$ may seem a little odd. An Arrow-Debreu model does not require an individual's endowment bundle to be actually in the consumption set. Instead, the condition states that the individual can get to the consumption set by disposing her endowments. This assumption is not entirely innocuous in that it implies that the individual can supply (if she desires) at least a little of every good to the market. For example, if $X_i = \mathbb{R}^L_+$, then this assumption would require the individual to be endowed with some of every good ($\omega_i \gg 0$). We can weaken this requirement to $\omega_i \geq \hat{x}_i$ for some $\hat{x}_i \in X_i$ if we are willing to accept a weaker notion of an equilibrium called a *quasiequilibrium*, which will be defined later.

The existence theorem given above is a central result in positive, or descriptive, economics, which seeks to describe and explain the "economy as it is." The two results that we give below are central results in welfare economics, which seeks to understand well-being at the aggregate level. Here, we touch upon normative economics where the main interest is in the desirability of economic outcomes, or the "economy as it ought to be." We do not really cross into normative economics, however, since our notion of desirability is that of efficiency, rather than a value-based criterion like "fairness" or "justice."

Definition 2.7. A feasible allocation (x, y) is *Pareto optimal* (or, *Pareto efficient*) if there is no other feasible allocation (x', y') such that $x'_i \succsim_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

□

If such an allocation (x', y') exists, it is said to *Pareto dominate*, or *Pareto improve*, (x, y). To put the definition differently, a feasible allocation (x, y) is Pareto optimal if any other allocation that has $x'_i \succ_i x_i$ for some i is either not feasible or has $x'_k \prec_k x_k$ for some k. That is, there is no way to make any one better off without making someone worse off. If an economy is stuck in a non-Pareto optimal allocation, it means that it is possible to improve someone's well-being without harming anyone but that the economy is unable to find a way to do it. Most reasonable people will agree that this is not a good outcome. Fortunately, Walrasian equilibrium is indeed Pareto optimal under very general conditions. In fact, this is true even when individuals are given lump-sum wealth transfers before they make their consumption decisions.

Let T_i be a wealth transfer to individual i. Individual i's budget set with transfers is

$$B_i(p, y, T_i) = \left\{ x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j + T_i \right\}.$$

Let $x_i(p, y, T_i)$ be individual i's demand correspondence.

Definition 2.8. In an Arrow Debreu economy, a *Walrasian (or, competitive) equilibrium with transfers* is a price vector p^*, an allocation (x^*, y^*), and wealth transfers $T = (T_1, ..., T_I)$ such that

1. For all firm j, $y_j^* \in y_j(p^*)$, (profit maximization)
2. For all individual i, $x_i^* \in x_i(p^*, y^*, T_i)$, (preference maximization)
3. $\sum_{i=1}^{I} x_i^* = \bar{\omega} + \sum_{j=1}^{J} y_j^*$, and (market clearance)
4. $\sum_{i=1}^{I} T_i = 0$. (balanced budget)

□

Note that a Walrasian equilibrium is simply a Walrasian equilibrium with transfers where transfers are zero for everyone. We are now ready to state the first welfare theorem.

Theorem 2.9 (First fundamental theorem of welfare)**.** *Suppose we have an Arrow Debreu economy in which all the preferences are locally non-satiated. If (x^*, y^*, p^*, T) is a Walrasian equilibrium with transfers, then the allocation (x^*, y^*) is Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.*

The first welfare theorem is a powerful result that captures in a mathematically rigorous way Adam Smith's notion of "invisible hand." It shows that a decentralized market, in which individuals are maximizing their own welfare and firms are maximizing their own profit without regard to others, nevertheless results in a socially desirable outcome: efficient allocation of available resources. Moreover, the result holds rather robustly. Besides the primitives of the model, such as complete market and price-taking assumption, the only substantive requirement in Theorem 2.9 is that the preferences are locally non-satiated. The second welfare theorem examines whether the converse holds. That is, it examines whether every efficient allocation can be brought about as a decentralized market outcome. As we will see, the answer is yes, by making appropriate income transfers. However, the result is less robust than the first welfare theorem and requires more conditions. In particular, it requires convexity. (We will examine why convexity is necessary later in the contexts of specific models.)

Theorem 2.10 (Second fundamental theorem of welfare)**.** *Suppose we have an Arrow Debreu economy in which all the consumption sets are* $\mathbb{R}_+$*, all the preferences are convex and strongly monotone, all the production sets are convex, and there are* $y_j \in Y_j$ *such that* $\bar{\omega} + \sum_j y_j \gg 0$*. Then for every Pareto optimal allocation* (x^*, y^*)*, there exists* $p^* \neq 0$ *and* T *such that* (x^*, y^*, p^*, T) *is a Walrasian equilibrium with transfers.*

Strictly speaking, not all the conditions in Theorem 2.10 are necessary. The required conditions can be weakened in several ways, and they will be discussed later when we return to this topic in greater detail.

3 Introductory Models

We now move away from the abstract discussions of the general model and present three specific examples of an Arrow-Debreu economy. We will investigate the central concepts of Walrasian equilibrium, Pareto optimality, and their relationship in these concrete contexts.

3.1 2 × 2 Pure exchange economy

In this economy, there are two individuals and two goods (hence 2 × 2), and no production takes place. Instead of production, the two individuals are born with some endowments of the goods and bring them to the market and trade with each other. This is an example of an Arrow-Debreu economy where,

- $L = 2$, $I = 2$, and $J = 0$.[2]
- Consumption space is $X_i = \mathbb{R}^2_+$. Individual i has preference $\succsim_i$ which is complete, transitive, and continuous and an endowment $\omega_i \geq 0$. Let $u_i(\cdot)$ denote the utility function representing $\succsim_i$. Assume that $\bar{\omega} \gg 0$,

Letting $L \geq 2$ and $I \geq 2$ in the above specification yields a more general pure exchange economy (without the qualifier 2 × 2).

3.1.1 Pareto optimality

An *allocation* in this economy is a vector $x = (x_1, x_2) = \big((x_{11}, x_{21}), (x_{12}, x_{22})\big) \in \mathbb{R}^4_+$ that specifies the amount of goods for each individual. A *feasible allocation* is an allocation that is an exact division of the aggregate endowment between the two individuals. That is, an allocation $\big((x_{11}, x_{21}), (x_{12}, x_{22})\big)$ satisfying $x_{11} + x_{12} = \bar{\omega}_1$ and $x_{21} + x_{22} = \bar{\omega}_2$. Since the sole economic activity in an exchange economy is trading and no good is created or destroyed, the individuals must always end up in a feasible allocation. The definition of Pareto optimality reduces to the following in the present case.

Definition 3.1. In a 2 × 2 exchange economy, feasible allocation $x = (x_1, x_2)$ is *Pareto optimal* (or *Pareto efficient*) if there is no other feasible allocation $x' = (x'_1, x'_2)$ satisfying $x'_i \succsim_i x_i$ for all i and $x'_i \succ_i x_i$ for some i. □

If there are no restrictions to trading, we should only observe Pareto optimal allocations as the end result of trading. Otherwise, there is a wasted trading opportunity.

[2] Some models of pure exchange economy allow free disposal. Those models would have $J = 1$, with $Y_1 = \mathbb{R}^L_-$.

The two individuals can trade some more to make someone better off without making the other person worse off. So, Pareto optimaliy is a sensible definition of an "efficient" outcome. To visualize Pareto optimality in a 2×2 economy, we employ a device called *Edgeworth box*.

As an example, suppose individual 1 has utility function $u_1(x_{11}, x_{21}) = x_{11}^{\frac{1}{2}} x_{21}^{\frac{1}{2}}$ and individual 2 has $u_2(x_{12}, x_{22}) = x_{12}^{\frac{1}{3}} x_{22}^{\frac{2}{3}}$. Both utility functions are Cobb-Douglas utilities and in particular represent preferences that are strongly monotone and strictly convex. Suppose the individuals start with endowments $\omega_1 = (1, 4)$ and $\omega_2 = (4, 3)$. The indifference curves passing through their endowments are illustrated in figure 3.1.

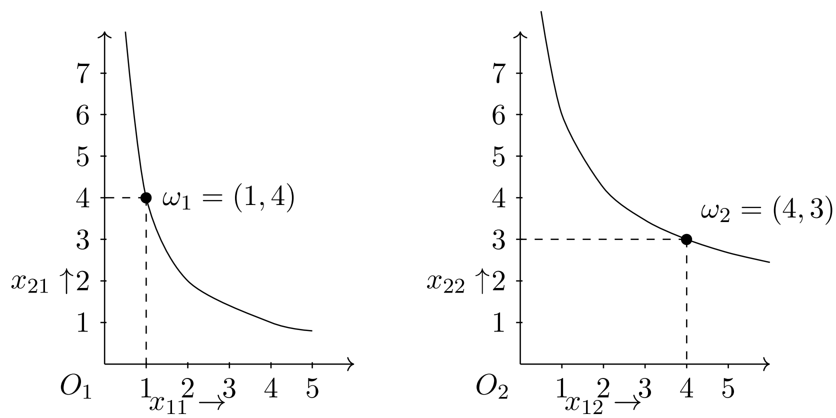

Figure 3.1: Indifference curves through the endowments.

To visualize the interaction between the two individuals, we merge the two graphs into a single diagram. First, rotate the graph of individual 2 by 180°, as in figure 3.2.

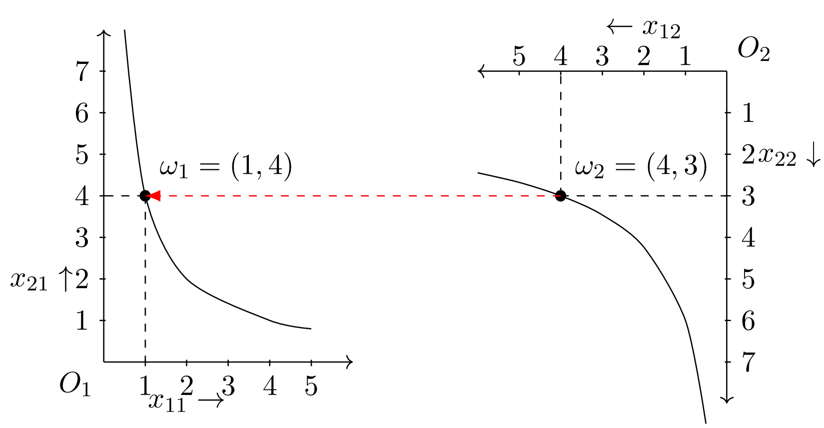

Figure 3.2: Axis for individual 2 rotated.

Next, slide the graph of individual 2 so that the endowment of individual 1 and the endowment of individual 2 meet (see figure 3.3).

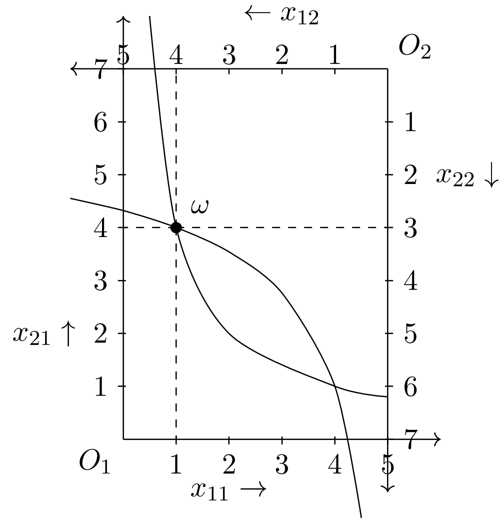

Figure 3.3: Axis for individual 2 rotated and merged.

Notice that the diagram now looks like a box. Hence, this is called an Edgeworth box, and a 2×2 pure exchange economy is often called an *Edgeworth box economy*. By making the endowment of the two individuals meet at a point, we have created a box whose length is equal to the total endowment of good 1 ($\bar{\omega}_1 = 5$) and the height is the total endowment of good 2 ($\bar{\omega}_2 = 7$). This turns every point in the box into a feasible allocation. That is, point (a, b) (measured from the bottom left corner O_1) in the box represents an allocation where individual 1 gets $x_1 = (a, b)$ (by counting from the bottom left corner) and individual 2 gets $x_2 = (\bar{\omega}_1 - a, \bar{\omega}_2 - b)$ (by counting from the top right corner).

The indifference curves in this example intersect each other (instead of being tangent) at the endowment allocation. Since the utility of individual 1 is increasing as we move northeast in the box and the utility of individual 2 is increasing as we move southwest, the endowment allocation is not Pareto optimal. For example, allocation A in figure 3.4 makes individual 2 to better off than her endowment while leaving individual 1 no worse off than her endowment. Of course, A is not Pareto optimal either since we can find another allocation, such as B, that makes individual 1 better off than A without making individual 2 worse off. Pareto optimality is not reached until we get to a point like C, where the upper contour sets of the two individuals' indifference curves (the areas of the consumption set containing bundles that are at least as good as those on the indifference curves) do not overlap.

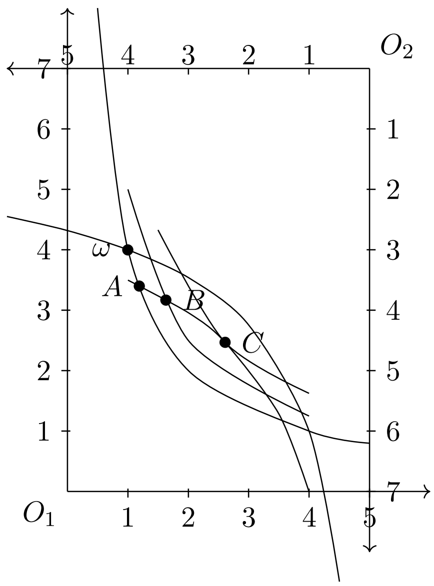

Figure 3.4: Pareto Optimality.

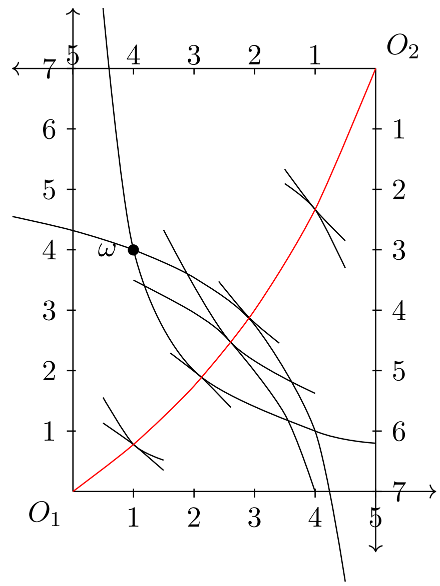

Figure 3.5: Pareto set.

In general, there are many Pareto optimal allocations in an economy, and the set of all such allocations is called the *Pareto set*. If preferences are strongly monotone, as in this example, the Pareto set will go through the two origins, O_1 and O_2. This is because at an origin, one individual has everything and moving from there to another feasible allocation necessarily means that she must give up some of at least one good, which will decrease her utility. In addition, all the Pareto optimal allocations in this example occur where indifference curves are tangent to each other. However, at the boundary of the Edgeworth box, "tangency" does not necessarily mean that the slopes of the indifference curves are equal, as figure 3.6 illustrates. Also, note that the indifference curves of the individuals in figure 3.6 do not actually extend into the negative areas (where $x_{22} < 0$). They would stop at where $x_{22} = 0$. Nevertheless, they were drawn this way to illustrate what the overlapping region would look like if the indifference curves could extend into the negative territory.

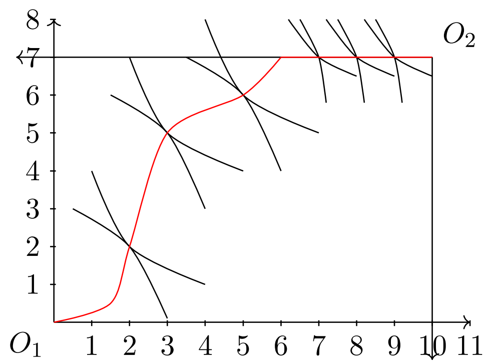

Figure 3.6: Pareto set that includes boundary allocations.

As we noted earlier, we should expect to see only Pareto optimal allocations as an outcome of trading if there are no barriers to trade. However, where in the Pareto set the economy ends up depends on other factors, such as bargaining ability of the individuals. Our example should also make it clear that Pareto optimality is a criterion for judging efficiency and not fairness. An allocation where one person has everything (assuming strong monotonicity) is Pareto optimal, though perhaps not fair. Of course, if trading is voluntary, then we expect the individuals to end up at least as well as what they started with. Such allocations are in the part of the Pareto set that lies between the indifference curves going through the individuals' endowments (the red section of the Pareto set in figure 3.7). Some authors call this area the *contract curve* while others call this the *core* and use contract curve to mean the entire Pareto set. We will use the term core in this note.

Definition 3.2. In a 2×2 exchange economy, *core* is the set of Pareto optimal allocations (x_1, x_2) satisfying $x_1 \succsim_1 \omega_1$ and $x_2 \succsim_2 \omega_2$. □

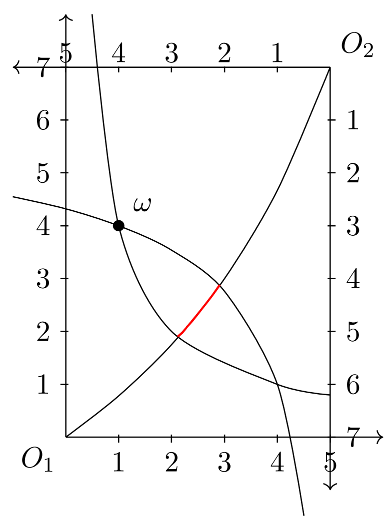

Figure 3.7: Core in a 2×2 exchange economy.

When the utility functions are differentiable, interior Pareto optimality can be characterized by the calculus tangency condition and the feasibility condition:

$$MRS_1 = \frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}} = MRS_2 \qquad \text{(tangency)}$$

$$x_{\ell 1} + x_{\ell 2} = \omega_{\ell 1} + \omega_{\ell 2} \text{ for } \ell = 1, 2. \qquad \text{(feasibility)}$$

Example 3.3 (Pareto optimality)**.** Suppose

$$u_1(x_{11}, x_{21}) = (x_{11})^{\frac{1}{2}}(x_{21})^{\frac{1}{2}} \qquad \omega_1 = (1, 4)$$

$$u_2(x_{12}, x_{22}) = (x_{12})^{\frac{1}{3}}(x_{22})^{\frac{2}{3}} \qquad \omega_2 = (4, 3)$$

To find the Pareto set, we solve $MRS_1 = MRS_2$:

$$\begin{aligned} MRS_1 &= \frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{\frac{1}{2}(x_{11})^{-\frac{1}{2}}(x_{21})^{\frac{1}{2}}}{\frac{1}{2}(x_{11})^{\frac{1}{2}}(x_{21})^{-\frac{1}{2}}} = \frac{x_{21}}{x_{11}} \\ MRS_2 &= \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}} = \frac{\frac{1}{3}(x_{12})^{-\frac{2}{3}}(x_{22})^{\frac{2}{3}}}{\frac{2}{3}(x_{12})^{\frac{1}{3}}(x_{22})^{-\frac{1}{3}}} = \frac{x_{22}}{2x_{12}} \\ \Longrightarrow \frac{x_{21}}{x_{11}} &= \frac{x_{22}}{2x_{12}}. \end{aligned}$$

Next, we use the feasibility conditions to eliminate two variables. Here, we will express everything in terms of x_{11}. Substituting $x_{11} + x_{12} = \bar{\omega}_1 = 5$ and $x_{21} + x_{22} = \bar{\omega}_2 = 7$, into the tangency condition yields

$$\begin{aligned} \frac{x_{21}}{x_{11}} = \frac{7 - x_{21}}{2(5 - x_{11})} \iff (10 - 2x_{11})x_{21} &= (7 - x_{21})x_{11} \\ 10x_{21} - 2x_{11}x_{21} &= 7x_{11} - x_{11}x_{21} \\ 10x_{21} - x_{11}x_{21} &= 7x_{11} \\ x_{21} &= \frac{7x_{11}}{10 - x_{11}}. \end{aligned}$$

Thus,

$$\text{Pareto set } = \left\{ \left(\left(x_{11}, \frac{7x_{11}}{10 - x_{11}} \right), \left(5 - x_{11}, 7 - \frac{7x_{11}}{10 - x_{11}} \right) \right) : x_{11} \in [0, 5] \right\}.$$

This is the Pareto set illustrated in figure 3.5. Some of the allocations in the set are

$$\big((0,0),(5,7)\big),\ \big((1,\tfrac{7}{9}),(4,6\tfrac{2}{9})\big),\ \big((2,\tfrac{14}{8}),(3,5\tfrac{2}{8}),\ \big((3,\tfrac{21}{7}),(2,4)\big),$$
$$\big((4,\tfrac{28}{6}),(1,2\tfrac{2}{6})\big),\ \text{and } \big((5,7),(0,0)\big).$$

□

3.1.2 Walrasian equilibrium

An important missing factor in the discussion of the Edgeworth box economy so far has been prices. Pareto set describes the set of outcomes we expect to see if any trade is possible. However, in most economies traders use prices to value goods when they trade. Thus, we now incorporate prices into the economy and ask what outcomes we should expect if individuals take prices as given and seeks to trade in a way that maximizes their utility.

An individual's utility maximization problem in the current setting is

$$\max_{x_{1i}, x_{2i}} \ u_i(x_{1i}, x_{2i}) \quad \text{s.t.} \quad p_1 x_{1i} + p_2 x_{2i} \le p_1 \omega_{1i} + p_2 \omega_{2i}.$$

Note that aside from the source of wealth being derived from the value of the endowment, this is the same utility maximization problem we have seen in consumer theory. The solution to the individual's utility maximization problem (Marshallian demand) is denoted by $x_i(p_1, p_2)$. Since the demand is homogeneous of degree zero in prices, we often standardize (also called normalizing) the scaling of the prices by setting $p_2 = 1$ or $p_1 + p_2 = 1$.

Remark. Price is a double-edged sword for the individuals in a general equilibrium context. If the price of a good goes up, it makes the good less affordable. At the same time, it can increase the value of their endowment and hence generate greater income. □

Example 3.4 (Utility Maximization)**.** Suppose the utility functions and the endowments of the individuals are given by

$$\begin{aligned} u_1(x_{11}, x_{21}) &= (x_{11})^{\frac{1}{2}}(x_{21})^{\frac{1}{2}} \qquad \omega_1 = (1, 4) \\ u_2(x_{12}, x_{22}) &= (x_{12})^{\frac{1}{3}}(x_{22})^{\frac{2}{3}} \qquad \omega_2 = (4, 3) \end{aligned}$$

Then the individuals' demand functions are

$$\begin{aligned} x_{11}(p_1, p_2) &= \frac{1}{2}\left(\frac{w_1}{p_1}\right) = \frac{p_1 + 4p_2}{2p_1} \quad \text{and} \quad x_{21}(p_1, p_2) = \frac{1}{2}\left(\frac{w_1}{p_2}\right) = \frac{p_1 + 4p_2}{2p_2} \\ x_{12}(p_1, p_2) &= \frac{1}{3}\left(\frac{w_2}{p_1}\right) = \frac{4p_1 + 3p_2}{3p_1} \quad \text{and} \quad x_{22}(p_1, p_2) = \frac{2}{3}\left(\frac{w_2}{p_2}\right) = \frac{2(4p_1 + 3p_2)}{3p_2}. \end{aligned}$$

Now, suppose the prices are $p_1 = 1.5$ and $p_2 = 1$. Then,

$$\begin{aligned} x_{11}(p_1, p_2) &= \frac{1.5 + 4}{3} = 1.83 \quad \text{and} \quad x_{21}(p_1, p_2) = \frac{1.5 + 4}{2} = 2.75 \\ x_{12}(p_1, p_2) &= \frac{6 + 3}{4.5} = 2 \quad \text{and} \quad x_{22}(p_1, p_2) = \frac{2(6 + 3)}{3} = 6. \end{aligned}$$

The individuals' utility maximization problems are depicted in figure 3.8. The slope of the price vector $p = (p_1, p_2)$ is $\frac{p_2}{p_1}$ while the slope of the budget line is $-\frac{p_1}{p_2}$. Thus, starting from the endowment of individual i, the budget line and the price vector are perpendicular. More generally, starting from any bundle (x_1, x_2) the iso-expenditure line through that bundle and the price vector are perpendicular. □

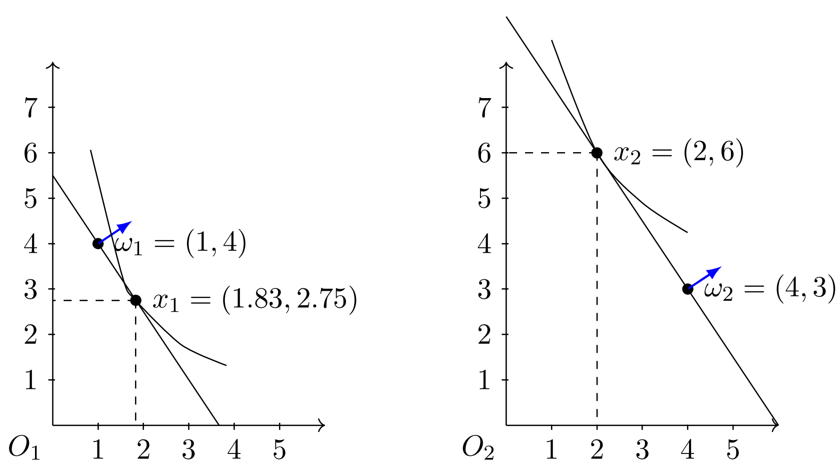

Figure 3.8: Utility maximization.

Examining the individuals' consumption choices separately, as we have done in the above example, makes it difficult to see whether they are compatible with one another or the fundamentals of the economy. Thus, we combine the two separate graphs into a single Edgeworth box diagram. First, rotate the axis of the individual 2 by 180°, which yields figure 3.9.

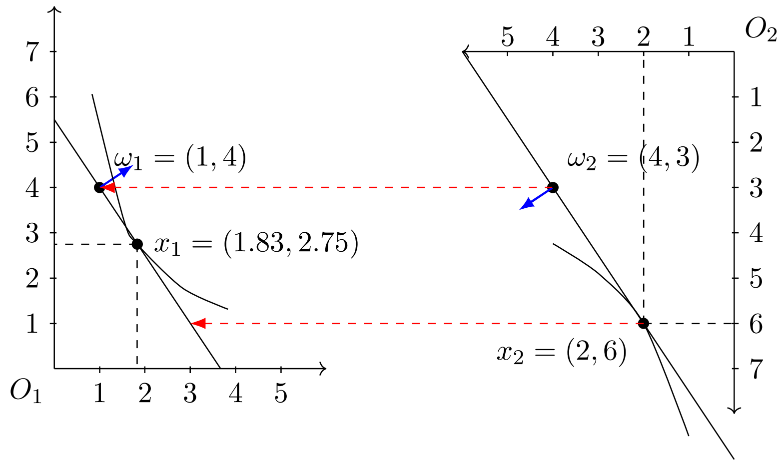

Figure 3.9: Utility maximization, axis for individual 2 rotated.

Sliding the graph until the endowments of the individuals coincide yields the Edgeworth box diagram shown in figure 3.10.

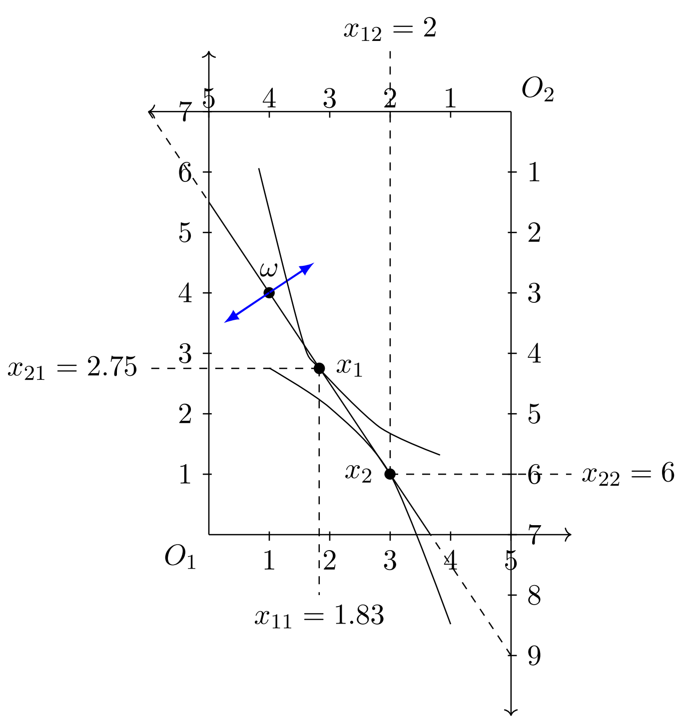

Figure 3.10: Edgeworth box, non-equilibrium.

As the figure shows, the price vector $p = (1.5, 1)$ does not balance the desires of the two individuals since they do not want to be at the same location in the diagram. In fact, in the picture,

$$\begin{aligned} x_{11}(p_1,p_2) + x_{12}(p_1,p_2) &= 1.83 + 2 < 5 = \bar{\omega}_1 \\ \text{and} \quad x_{21}(p_1,p_2) + x_{22}(p_1,p_2) &= 2.75 + 6 > 7 = \bar{\omega}_2. \end{aligned}$$

So there is an *excess supply* of good 1 and *excess demand* for good 2. Thus, we expect that prices will adjust. In particular, it seems that the price of good 1 should fall or the price of good 2 should rise (or more accurately, the price of good 1 relative to the price of good 2 should fall) to balance the desires of the individuals. Exactly how this may be accomplished, if at all, will be discussed later. For now, we will accept that when all is settled, the economy should be in a Walrasian equilibrium, whose definition reduces to the following in an Edgeworth box economy.

Definition 3.5. In a 2×2 exchange economy, *Walrasian (competitive) equilibrium* is a price vector $p^* = (p_1^*, p_2^*)$ and an allocation $x^* = (x_1^*, x_2^*)$ such that

1. For all individual i, $x_i^* \in x_i(p^*)$, and (preference maximization)
2. For all good ℓ, $x_{\ell 1}^* + x_{\ell 2}^* = \bar{\omega}_\ell$. (market clearance)

□

That is, Walrasian equilbrium is a situation where individuals are consuming their utility maximizing bundle and the markets clear. When individuals' utility maximizing bundles are unique, their demands are functions, and we can define the *market excess demand function*

$$z(p) = \sum_i x_i(p) - \bar{\omega}.$$

Then an equilibrium price vector is p^* such that $z(p^*) = 0$, meaning the prices at which the excess demand is zero in all the markets. This is a two-dimensional (one for each good) vector equation in two variables (the prices), which written as column vectors yields

$$\begin{bmatrix} z_1(p_1,p_2) \\ z_2(p_1,p_2) \end{bmatrix} = \begin{bmatrix} x_{11}(p_1,p_2) + x_{12}(p_1,p_2) \\ x_{21}(p_1,p_2) + x_{22}(p_1,p_2) \end{bmatrix} - \begin{bmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since only relative prices matter, we can normalize and remove one of the variables. For example, setting $p_2 = 1$ (and simplifying the equation a little further) yields

$$\begin{bmatrix} x_{11}(p_1,1) + x_{12}(p_1,1) - \bar{\omega}_1 \\ x_{21}(p_1,1) + x_{22}(p_1,1) - \bar{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, we have two equations with only one unknown, which in general could be problematic. However, this typically does not happen in practice because the two equations are not independent in most practical settings. For example, when preferences are strongly monotone, we have *Walras' Law* (see exercise), which yields the following.

$$\begin{aligned} p \bullet z(p) = 0 &\iff (p_1, p_2) \bullet \big(z_1(p_1, p_2), z_2(p_1, p_2)\big) \\ &\implies p_1 z_1(p_1, 1) + z_2(p_1, 1) = 0 \\ &\implies z_2(p_1, 1) = -p_1 z_1(p_1, 1). \end{aligned}$$

Since $p_1 > 0$ (see exercise), $z_2(p_1, 1) = 0$ if and only if $z_1(p_1, 1) = 0$. The following summarizes the result for more general exchange economies.

Theorem 3.6. *In a L-goods exchange economy with locally non-satiated preferences, suppose price vector p, with $p_\ell \neq 0$ for all ℓ, clears $L-1$ markets. Then it clears the last remaining market.*

The following gives a numerical example for finding a Walrasian equilibrium.

Example 3.7 (Walrasian equilibrium)**.** Continuing Example 3.4, recall that the economy was specified by

$$\begin{aligned} u_1(x_{11}, x_{21}) &= (x_{11})^{\frac{1}{2}}(x_{21})^{\frac{1}{2}} \qquad \omega_1 = (1, 4) \\ u_2(x_{12}, x_{22}) &= (x_{12})^{\frac{1}{3}}(x_{22})^{\frac{2}{3}} \qquad \omega_2 = (4, 3), \end{aligned}$$

which yielded demand functions

$$\begin{aligned} x_{11}(p_1, p_2) &= \frac{p_1 + 4p_2}{2p_1} \quad \text{and} \quad x_{21}(p_1, p_2) = \frac{p_1 + 4p_2}{2p_2} \\ x_{12}(p_1, p_2) &= \frac{4p_1 + 3p_2}{3p_1} \quad \text{and} \quad x_{22}(p_1, p_2) = \frac{2(4p_1 + 3p_2)}{3p_2}. \end{aligned}$$

To find the equilibrium price, we normalize $p_2^* = 1$ and equilibriate one of the two markets, say market for good 2:

$$\begin{aligned} x_{21}(p_1^*, p_2^*) + x_{22}(p_1^*, p_2^*) &= \frac{p_1^* + 4}{2} + \frac{2(4p_1^* + 3)}{3} = 7 = \bar{\omega}_2 \\ 3p_1^* + 12 + 16p_1^* + 12 &= 42 \\ p_1^* &= \frac{18}{19}. \end{aligned}$$

So the equilibrium price is $p^* = (\frac{18}{19}, 1)$ and the equilibrium allocation is

$$\begin{aligned} x_{11}^* &= \frac{\frac{18}{19} + 4}{2(\frac{18}{19})} = \frac{4.95}{1.89} = 2.61 \quad \text{and} \quad x_{21}^* = \frac{\frac{18}{19} + 4}{2} = \frac{4.95}{2} = 2.47 \\ x_{12}^* &= \frac{4\frac{18}{19} + 3}{3(\frac{18}{19})} = \frac{6.79}{2.84} = 2.39 \quad \text{and} \quad x_{22}^* = = \frac{2(4\frac{18}{19} + 3)}{3} = \frac{13.58}{3} = 4.53. \end{aligned}$$

To check that markets clear:

$$\begin{aligned} x_{11}^* + x_{12}^* &= 2.61 + 2.39 = 5 = \bar{\omega}_1. \\ x_{21}^* + x_{22}^* &= 2.47 + 4.53 = 7 = \bar{\omega}_2. \end{aligned}$$

The equilibrium is illustrated in figure 3.11. □

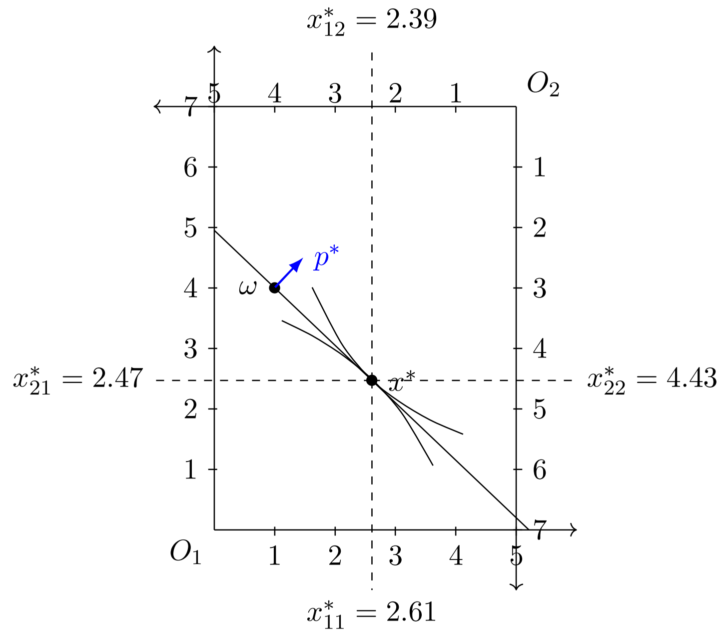

Figure 3.11: Edgeworth box, equilibrium.

We have already seen the existence theorem for Walrasian equilibrium in a general Arrow-Debreu economy. The following gives a simple version of the existence theorem for an Edgeworth box economy.

Theorem 3.8. *Suppose every individual in a 2×2 exchange economy has a complete, transitive, continuous, strongly monotone, and strictly convex preference and a strictly positive endowment. Then a Walrasian equilibrium exists.*

To illustrate why an equilibrium must exist under the stated assumptions, we can use the individuals' *offer curves*, which are curves that trace the individuals' demands as prices vary. Let's call the individuals' indifference curves corresponding to their initial endowments their endowment indifference curves. Note that if the endowment allocation is Pareto optimal, then it is a Walrasian equilibrium with the equilibrium price ratio ($\frac{p_1}{p_2}$) given by the slope of one of the endowment indifference curves. Thus, it is enough to consider cases where the endowment allocation is not Pareto optimal. Under the assumptions on the preferences, each individual's demand is a continuous function that is defined on positive prices. This implies that their offer curves are continuous and tangent to their respective endowment indifference curves. They also lie on the upper contour sets of the endowment indifference curves (the part of the consumption set that represent bundles that are at least as good as the ones on the indifference curve), as shown in figure 3.12. (In the figure, blue curve is individual 1's offer curve and the red curve is individual 2's). Moreover, strong monotonicity means that each individual's demand for good 1 must go to infinity as price of good 1 goes to zero, and similarly the demand for good 2 must go to infinity as the price of good 2 goes to zero. Thus, the individuals' offer curves must intersect at some point other than the endowment. That intersection point is a Walrasian equilibrium allocation.

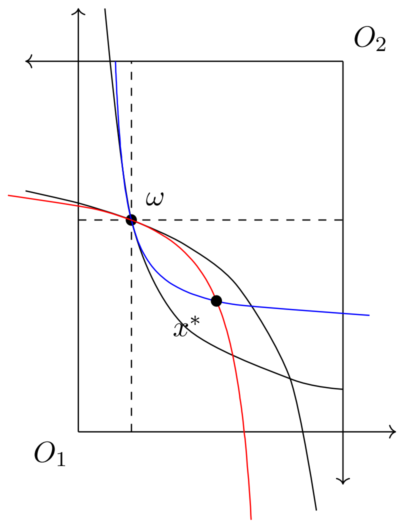

Figure 3.12: Offer curves and the existence of Walrasian equilibrium.

We should emphasize again that Theorem 3.8, as well as Theorem 2.6, gives the sufficient condition for the existence of an equilibrium. It does not indicate that preferences must have these properties for the equilibrium to exist. In addition, the theorem guarantees that at least one equilibrium exists. It does not imply that it is unique. Figure 3.13 provides an example of an economy with multiple equilibria. In addition, figure 3.10 that illustrated a non-equilibrium situation had x_1 and x_2 on the "same side" of the endowment allocation on the budget line. In general, this need not be. Where the optimal demands are located on the budget line depends on the individuals' preferences, and they could be on the opposite sides of the endowment point, as figure 3.14 illustrates.

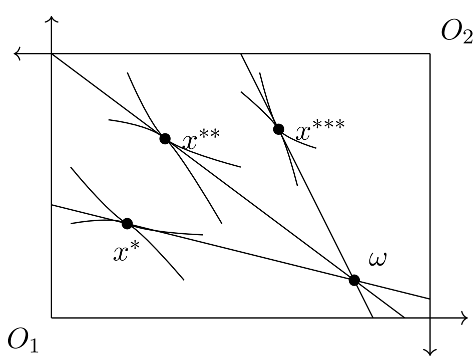

Figure 3.13: Multiple equilibria.

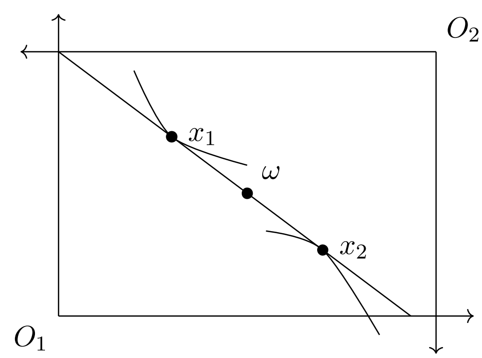

Figure 3.14: Demands on opposite sides of endowment allocation.

3.1.3 Welfare properties

We now relate Walrasian equilibrium to Pareto optimality. Reducing the first welfare theorem to the Edgeworth box economy setting yields the following.

Theorem 3.9. *Suppose every individual in a 2×2 exchange economy has a locally non-satiated preference. Then Walrasian equilibrium allocation is Pareto optimal.*

This is easy to see in the case of an interior equilibrium, where everyone is consuming positive amount of each good, since it occurs where the indifference curves of the two individuals are tangent to each other at the budget line. Therefore, an interior Walrasian equilibrium is always Pareto optimal. For example, when utility functions are differentiable, an interior equilibrium must satisfy

$$\frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{p_1}{p_2} = \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}}.$$

So clearly, we must have

$$\frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}},$$

which is the condition for Pareto optimality. To see why local non-satiation condition is needed, consider figure 3.15. In the figure, individual 1 has a thick indifference curve represented by the red area (which violates local non-satiation) while individual 2 has the standard strongly monotone, strictly convex preference. Price vector p^* and allocatin x^* constitute a Walrasian equilibrium. However, it is not Pareto optimal because $\hat{x}$ makes individual 2 better off than x^* without making individual 1 worse off.

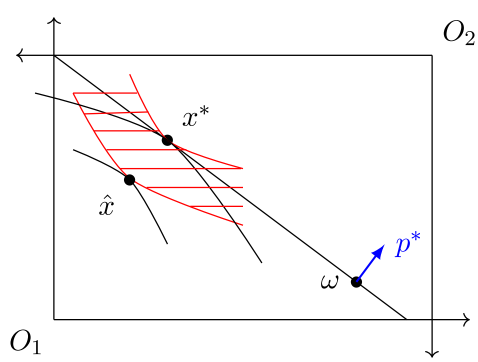

Figure 3.15: Non-Pareto optimal Walrasian equilibrium.

The importance of the first welfare theorem has been discussed already. The extent to which its converse holds (that is, whether every Pareto optimal allocation is an outcome of some Walrasian equilibrium) is the subject of the second welfare theorem. Even without thinking about it deeply, it is easy to see that the converse need not hold in general since the condition for an equilibrium is more restrictive than the condition for Pareto optimality. Figure 3.16 illustrates this graphically. In the figure, allocation $\hat{x} = (\hat{x}_1, \hat{x}_2)$ is Pareto optimal. However, if $\hat{x}$ is going to be an equilibrium allocation, it must be on the budget line that connects $\hat{x}$ and the endowment allocation ω. This means that the candidate for an equilibrium price vector must be p'. But, as we have drawn the figure, neither individual has $\hat{x}_i$ as their utility maximizing bundle at these prices. Thus, $\hat{x}$ cannot be an equilibrium allocation.

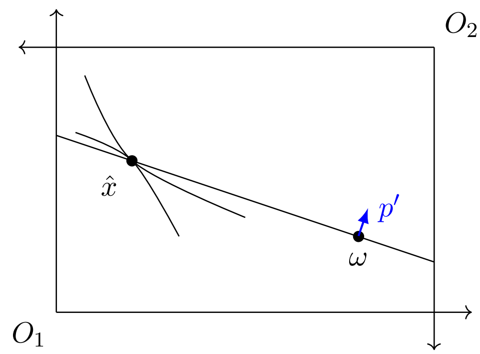

Figure 3.16: Non-equilibrium Pareto optimal allocation.

Note, however, that if the budget line happens to look like the one pictured in figure 3.17 below, then $\hat{x}$ will be an equilibrium. This suggests that if a central planner or a governmental authority wants to achieve, or *support*, $\hat{x}$ as an equilibrium outcome, it can first transfer the endowments between the two individuals so that they end up with $\hat{\omega}$, and then let them trade among themselves. However, governments typically cannot redistribute goods directly, as logistics of transferring goods are difficult.[3] Individuals also tend to get upset if a government comes and takes, for example, their car and gives it to someone else. They do however seem to mind somewhat less if the government collects money from them (e.g., through tax) and gives it to someone else (e.g., as welfare payment). Thus, we ask whether it is possible for the government to turn $\hat{x}$ into an equilibrium by making *monetary* transfers. It is easy to see that the answer is yes. The government simply needs to make transfers equal to the value of the endowment transfers at price $\hat{p}$. This is the idea behind the second welfare theorem that was provided earlier for Arrow-Debreu economies. We provide a simpler definition of an equilibrium with transfers and the second welfare theorem for the current setting.

[3]In addition, if the government can transfer the endowments, then it can simply allocate $\hat{x}$ to the individuals directly rather than bothering with $\hat{\omega}$.

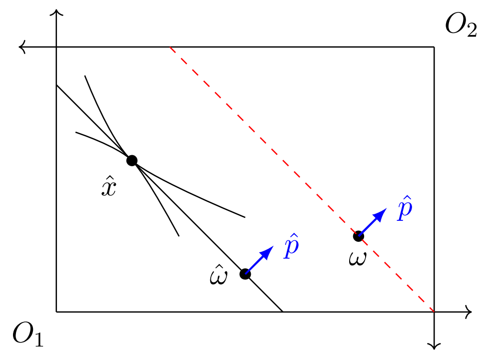

Figure 3.17: Walrasian equilibrium with transfers.

Definition 3.10. An allocation $\hat{x} = (\hat{x}_1, \hat{x}_2)$, price vector $\hat{p} = (\hat{p}_1, \hat{p}_2)$, and (monetary) transfers T_1 and T_2 together form a *Walrasian equilibrium with transfers* if

1. For all individual i, $\hat{x}_i$ is her utility maximizing bundle when her income is $\hat{p}_1\omega_{1i} + \hat{p}_2\omega_{2i} + T_i$.
2. $\hat{x}_{11} + \hat{x}_{12} = \bar{\omega}_1$ and $\hat{x}_{21} + \hat{x}_{22} = \bar{\omega}_2$, and
3. $T_1 + T_2 = 0$. □

Theorem 3.11. *Suppose every individual in a 2×2 exchange economy has a strictly convex and strongly monotone preference. Then every Pareto optimal allocation can be supported as a Walrasian equilibrium with transfers.*

The first fundamental theorem of welfare states that a government, or a central planner, has no efficiency justification for intervening in the market since the market solution, the Walrasian equilibrium, is always efficient. Because of this, this theorem has been cited by those who call for limited role for the government in the economy. However, the second fundamental theorem of welfare provides an equally powerful counter-perspective to such view. It states that, any Pareto optimal allocation can be achieved as a market outcome once proper income transfers are made. Thus, should a government wish to intervene in a market to achieve a distributional objective, it can do so without losing any efficiency. (An important caveat to this is that transfers are lump-sum and not tied to any individual's actions, which is not how most government transfers work).

Example 3.12. Continuing the earlier example, suppose we again have

$$u_1(x_{11}, x_{21}) = (x_{11})^{\frac{1}{2}}(x_{21})^{\frac{1}{2}} \qquad \omega_1 = (1, 4)$$
$$u_2(x_{12}, x_{22}) = (x_{12})^{\frac{1}{3}}(x_{22})^{\frac{2}{3}} \qquad \omega_2 = (4, 3).$$

The Walrasian equilibrium is $p^* = \left(\frac{18}{19}, 1\right)$ and $x^* = \big((2.61, 2.47), (2.39, 4.53)\big)$. Suppose the central planner wants to obtain $\hat{x} = (\hat{x}_1, \hat{x}_2)$, where $\hat{x}_1 = (4, \frac{14}{3})$ and $\hat{x}_2 = (1, \frac{7}{3})$ as an outcome instead. In the following, we show that it is possible to obtain this as a decentralized outcome by making appropriate transfers.

First we verify that $\hat{x}$ is Pareto optimal by checking the marginal rates of substitution at $\hat{x}$.

$$MRS_1\Big|_{(4,\frac{14}{3})} = \frac{x_{21}}{x_{11}}\Big|_{(4,\frac{14}{3})} = \frac{\frac{14}{3}}{4} = \frac{7}{6}.$$
$$MRS_2\Big|_{(1,\frac{7}{3})} = \frac{x_{22}}{2x_{12}}\Big|_{(1,\frac{7}{3})} = \frac{\frac{7}{3}}{2(1)} = \frac{7}{6}.$$

Since the marginal rates of substitution are the same, the indifference curves are indeed tangent at $\hat{x}$, meaning it is Pareto optimal. So the second welfare theorem guarantees that $\hat{x}$ can be supported as an equilibrium with transfers. The supporting price ratio must be equal to the marginal rate of substitution, so $\frac{\hat{p}_1}{\hat{p}_2} = \frac{7}{6}$. Keeping with our normalization, we set $\hat{p}_2 = 1$, which means $\hat{p} = (\frac{7}{6}, 1)$. Next, we find T_1 by calculating how much must be given to individual 1 to make $\hat{x}_1$ be on her budget line. That is, T_1 satisfies

$$\begin{aligned}
\hat{p}_1\hat{x}_{11} + \hat{p}_2\hat{x}_{21} &= \hat{p}_1\omega_{11} + \hat{p}_2\omega_{21} + T_1 \\
\Longrightarrow T_1 &= \hat{p}_1(\hat{x}_{11} - \omega_{11}) + \hat{p}_2(\hat{x}_{21} - \omega_{21}) = \tfrac{7}{6}(4-1) + 1(\tfrac{14}{3} - 4) \\
&= \tfrac{7}{2} + \tfrac{2}{3} = \tfrac{25}{6} \\
\Longrightarrow T_2 &= \hat{p}_1(\hat{x}_{12} - \omega_{12}) + \hat{p}_2(\hat{x}_{22} - \omega_{22}) = \tfrac{7}{6}(1-4) + 1(\tfrac{7}{3} - 3) \\
&= -\tfrac{7}{2} - \tfrac{2}{3} = -\tfrac{25}{6}.
\end{aligned}$$

That is, if we take $\$\frac{25}{6}$ from individual 2 and transfer it to individual 1, and then leave the economy alone, it should find $\hat{p} = (\frac{7}{6}, 1)$ as the equilibrium price, and the individuals will choose $\hat{x}_1 = (4, \frac{14}{3})$ and $\hat{x}_2 = (1, \frac{7}{3})$. Formally, we say that we have shown that allocation $\hat{x} = \big((4, \frac{14}{3}), (1, \frac{7}{3})\big)$ can be supported as a Walrasian equilibrium with transfers. □

3.2 Robinson Crusoe economy

The previous model had no production. To study production in a simplest possible setting, we now consider an economy with one individual, one firm, and two goods. Since there is only one individual (hence the moniker Robinson Crusoe), the same individual is both the consumer and the producer (as the owner of the firm). However, we assume that the individual's decisions as the consumer is made independently from her decisions as the owner of the firm. In particular, she acts as a price taker in both roles. This may seem strange in the current setting but is necessary to have a competitive market and will become more palatable when we extend the model to an economy with many individuals and many firms.

Robinson Crusoe economy is a special case of Arrow-Debreu economies, where $L = 2$, $I = 1$, and $J = 1$. However, it has its own set of notations that have carried over from single output producer theory, which is also used here. The setting of the economy as follows.

- $L = 2$. There are two goods, denoted $\ell = 1, 2$. For concreteness, we will assume that good 1 is leisure (measured in units of time) and good 2 is an ordinary consumption good (e.g., food).

- $I = 1$. There is one individual, with the same consumption space and preference as in the 2×2 exchange economy setting. The individual's consumption bundle is denoted $x = (x_1, x_2)$, and her endowment bundle is $\omega = (\omega_1, \omega_2)$. We will typically assume $\omega = (\bar{L}, 0)$, where $\bar{L}$ is the total amount time available to the individual.

- $J = 1$. There is one firm, which takes labor as input and transforms it into the consumption good using an increasing and strictly concave production function $f(z)$, where z is the amount of labor input (measured in units of time). Note that leisure and labor are different uses of time. Leisure is time consumed in pleasurable activity and labor is time used in production.

- $w =$ price of labor (and also leisure), and $p =$ price of the consumption good. Note that p in the current setting is a price of a single good and not a vector of prices. In addition, w here is "double u"(for wage) not "omega" (ω) that is being used to denote endowment.

Firm's objective in this economy is to maximize profit. Thus, it solves

$$\max_z \; pf(z) - wz.$$

Let $z(w, p)$ be the solution to the firm's profit maximization problem, which is called the *(unconditional) input demand.* The profit maximizing output level $y(w, p) = f(z(w, p))$ is called the *supply function*, and the value function $\pi(w, p) = py(w, p) - wz(w, p)$, which gives the maximized profit level, is called the *profit function.* Some authors reverse the order of the variables in the notation (e.g., use $z(p, w)$ instead). Nevertheless, in a Robinson Crusoe economy w will always mean the price of labor (leisure) and p will mean the price of the output good, regardless of the order in which they appear.

To visualize the firm's production decisions, it is more convenient to describe a firm's production plan by a netput vector, in which the amount of input good being used is denoted by a negative number and the amount of output good being produced is denoted by a positive number. The firm's production set gives all the production plans it can choose, which assuming free disposal, is

$$Y = \big\{(-z, y) : z \geq 0 \text{ and } y \leq f(z)\big\}.$$

As an example, suppose $f(z) = 2z^{\frac{1}{2}}$. Then

$$Y = \big\{(-z, y) : z \geq 0 \text{ and } y \leq 2z^{\frac{1}{2}}\big\}.$$

The graph of $f(z)$ and the corresponding production set Y are illustrated in figure 3.18. As the figure shows, a production set is obtained by reflecting the graph of the production function about the vertical axis.

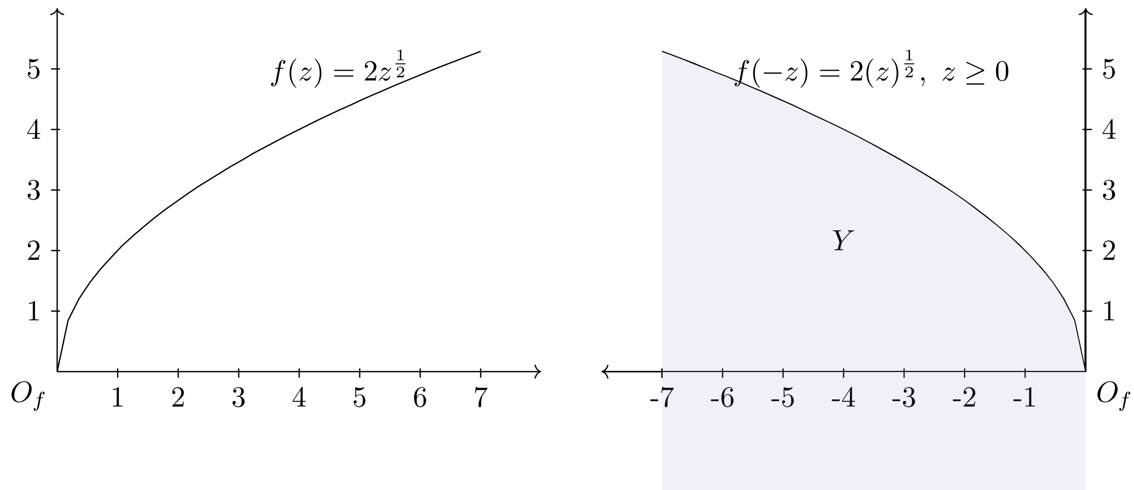

Figure 3.18: Production function and production set.

Using the production set, the firm's profit maximization problem can be described alternatively as

$$\max_{(-z,y)\in Y} \ (w,p)\bullet(-z,y).$$

To visualize the firm's profit maximization problem, recall that isoprofit lines are perpendicular to the price vector (w,p). Therefore, the profit maximizing production plan occurs at where an isoprofit line is tangent to the boundary of the production set, as seen in figure 3.19.

We assume that this is a private ownership economy, so the profit that the firm makes goes to its owners. Since there is only one individual in this economy, that individual is assumed to be the sole owner of the firm. If the individual consumes x_1 amount of leisure, she works $\bar{L} - x_1$ units of time and earns labor income $w(\bar{L} - x_1)$, which she can use to buy the consumption good. Thus, her utility maximization problem is

$$\max_{x_1,x_2} \ u(x_1,x_2) \quad \text{s.t.} \quad px_2 \leq w(\bar{L} - x_1) + \pi(w,p).$$

Or, equivalently,

$$\max_{x_1,x_2} \ u(x_1,x_2) \quad \text{s.t.} \quad wx_1 + px_2 \leq w\bar{L} + \pi(w,p).$$

Let $x_1(w,p)$ and $x_2(w,p)$ be the solution to the utility maximization problem. Individual's utility maximization problem is illustrated in figure 3.20.

Example 3.13. Consider a Robinson Crusoe economy in which the production function of the firm in a Robinson Crusoe economy is $f(z) = 2z^{\frac{1}{2}}$ and the individual's preference and endowments are

$$u(x_1,x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} \quad \text{and} \quad \bar{L} = 24.$$

We will find the firm's profit maximizing production plan and the individual's utility maximizing plan when the prices are $(w,p) = (\frac{1}{2}, 1)$.

The firm's profit maximization problem is:

$$\max_{z} \; p\left(2z^{\frac{1}{2}}\right) - wz$$

Solving the first order condition yields

$$pz^{-\frac{1}{2}} - w = 0 \implies z^{-\frac{1}{2}} = \frac{w}{p} \implies z(w,p) = \left(\frac{p}{w}\right)^2 = \left(\frac{1}{\frac{1}{2}}\right)^2 = 4.$$

$$\implies y(w,p) = 2z(w,p)^{\frac{1}{2}} = \frac{2p}{w} = \frac{2}{\frac{1}{2}} = 4.$$

$$\pi(w,p) = p\left(\frac{2p}{w}\right) - w\left(\frac{p}{w}\right)^2 = \frac{p^2}{w} = \frac{1}{\frac{1}{2}} = 2.$$

The individual's utility maximization problem is:

$$\max_{x_1,x_2} \; x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} \quad \text{s.t.} \quad wx_1 + px_2 \le w\bar{L} + \frac{p^2}{w}.$$

Since the individual has a Cobb-Douglas utility function, her demand is given by

$$x_1(w,p) = \frac{\text{income}}{3w} = \frac{w\bar{L} + \frac{p^2}{w}}{3w} = \frac{\frac{1}{2}(24) + \frac{1}{\frac{1}{2}}}{3(\frac{1}{2})} = (12+2)\frac{2}{3} = \frac{28}{3} = 9.333$$

$$x_2(w,p) = \frac{2 \text{ income}}{3p} = \frac{2\left(w\bar{L} + \frac{p^2}{w}\right)}{3p} = \frac{2\left(\frac{1}{2}(24) + \frac{1}{\frac{1}{2}}\right)}{3} = \frac{2}{3}(12+2) = \frac{28}{3}.$$

□

The following two figures illustrate the firm's profit maximization and the individual's utility maximization problem in Example 3.13.

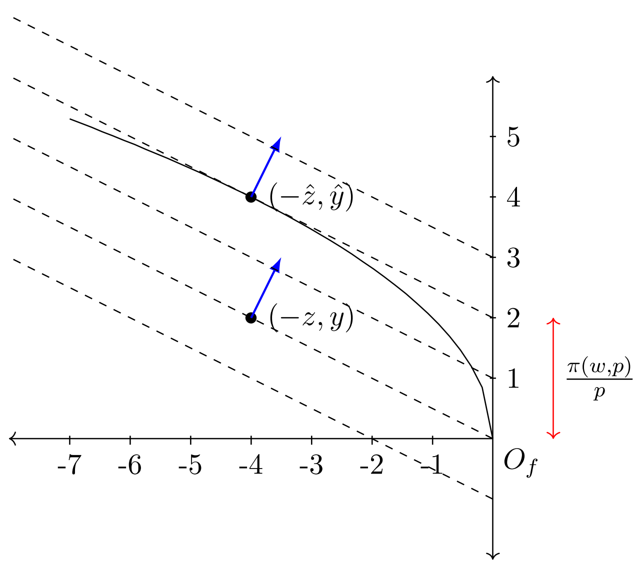

Figure 3.19: Profit maximization: $f(z) = 2z^{\frac{1}{2}}$, $(w,p) = (\frac{1}{2}, 1)$.

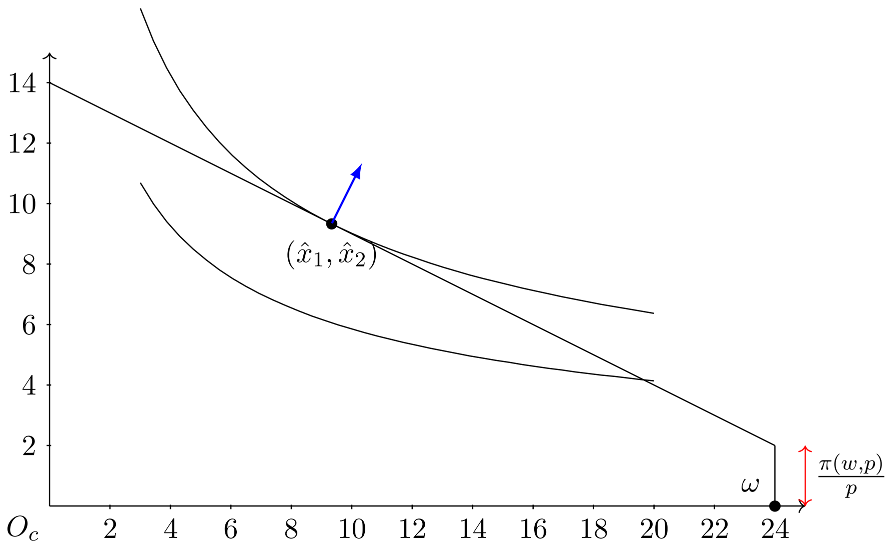

Figure 3.20: Utility maximization: $u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$, $(w, p) = (\frac{1}{2}, 1)$.

Looking at the firm's and the individual's problem separately makes it difficult to tell whether their choices are compatible with each other. Thus, we merge the two graphs together as we have done for an Edgeworth box economy. In the case of a Robinson Crusoe economy, we first place the graphs next to each other, as in figure 3.21.

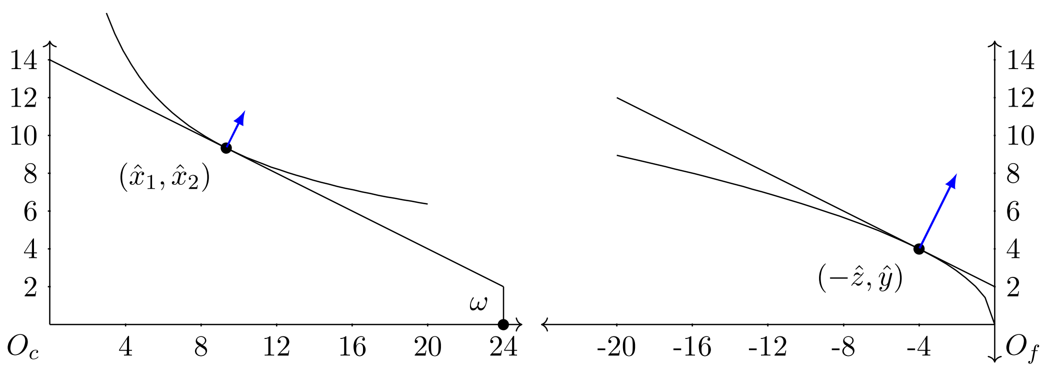

Figure 3.21: Individual's decision and the Firm's decision.

Then we slide the axis for the firm's graph until the firm's origin coincides with the endowment point of the consumer, as in figure 3.22 below.

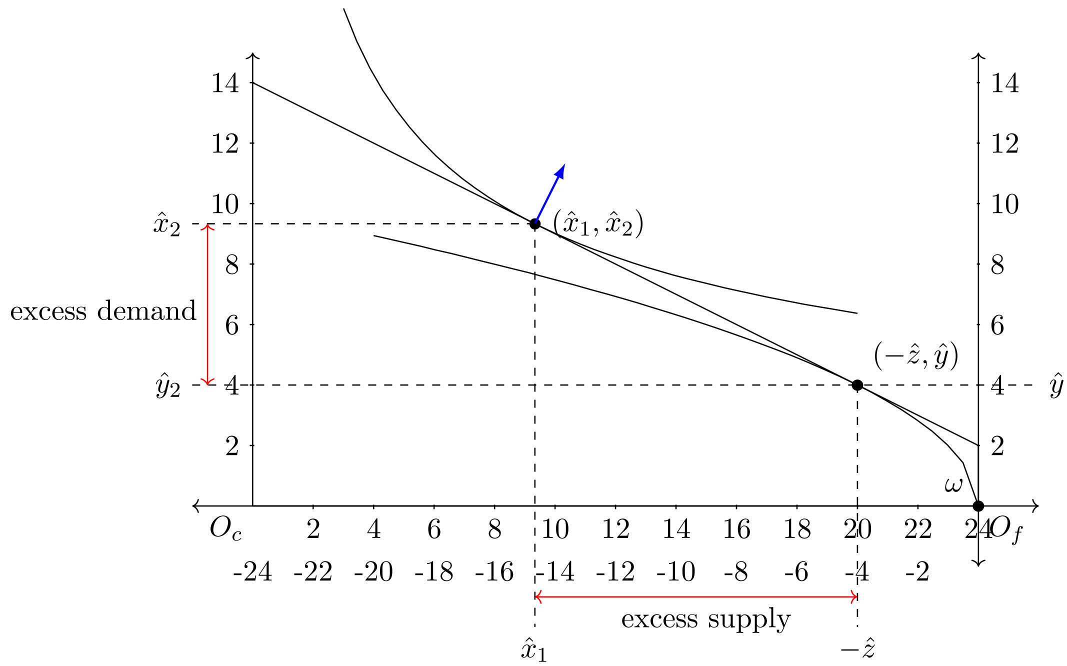

Figure 3.22: Robinson Crusoe economy: non-equilibrium.

Notice that in the figure, the individual's consumption desire is not compatible with the firm's production plan. Hence, this figure depicts an unstable situation. When the desires of the individual and the firm match, we have an equilibrium.

Definition 3.14. In a Robinson Crusoe economy, *Walrasian (competitive) equilibrium* is a price vector $p^* = (p_1^*, p_2^*)$ and an allocation $\big((x_1^*, x_2^*), (z^*, y^*)\big)$ such that

1. (z^*, y^*) is profit maximizing at prices (w^*, p^*), (π-max)
2. (x_1^*, x_2^*) is utility maximizing at prices (w^*, p^*), and (u-max)
3. $z^* + x_1^* = \omega_1$ and $x_2^* = \omega_2 + y^*$. (market clearance)

□

An illustration of a Walrasian equilibrium is given in figure 3.23. Notice that, like the Edgeworth box economy, one market will clears in this economy if and only if the other market clears.

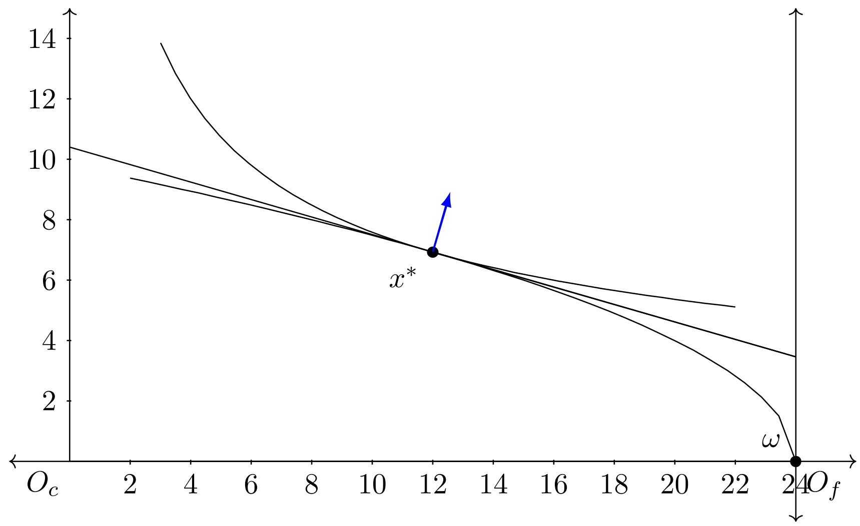

Figure 3.23: Robinson Crusoe economy: equilibrium.

Example 3.15. We find the Walrasian equilibrium of the economy in Example 3.13. The firm's input demand and supply functions were found to be

$$z(w,p) = \left(\frac{p}{w}\right)^2 \quad \text{and} \quad y(w,p) = \frac{2p}{w},$$

and the individual's demand functions were

$$x_1(w,p) = \frac{w\bar{L} + \frac{p^2}{w}}{3w} \quad \text{and} \quad x_2(w,p) = \frac{2\left(w\bar{L} + \frac{p^2}{w}\right)}{3p}.$$

To find the Walrasian equilibrium, we solve the market clearing condition. Substituting in $\bar{L} = 24$ and normalization $p = 1$ into the market clearing condition for the consumption good yields

$$x_2(w,p) = \frac{48w + \frac{2}{w}}{3} = \frac{2}{w} = y(w,p)$$
$$48w + \frac{2}{w} = \frac{6}{w} \implies 48w = \frac{4}{w} \implies w = \left(\frac{1}{12}\right)^{\frac{1}{2}} = 0.289.$$

Therefore, the Walrasian equilbrium price is $(w^*, p^*) = \left(\sqrt{\frac{1}{12}}, 1\right)$. The firm's profit (which depends on the normalization used) is

$$\pi(w^*, p^*) = \frac{1}{\frac{1}{\sqrt{12}}} = \sqrt{12} = 3.464.$$

The equilibrium allocation can be found by evaluating the individual's demand function and the firm's labor demand and supply functions at the equilibrium price.

$$z(w^*, p^*) = \left(\frac{p}{w}\right)^2 = \left(\frac{1}{\frac{1}{\sqrt{12}}}\right)^2 = 12$$

$$y(w^*, p^*) = \frac{2p}{w} = \frac{2}{\frac{1}{\sqrt{12}}} = 2\sqrt{12} = 6.928$$

$$x_1(w^*, p^*) = \frac{w\bar{L} + \frac{p^2}{w}}{3w} = \frac{\frac{1}{\sqrt{12}}24 + \frac{1}{\frac{1}{\sqrt{12}}}}{3\frac{1}{\sqrt{12}}} = \frac{24 + 12}{3} = 12$$

$$x_2(w^*, p^*) = \frac{2\left(w\bar{L} + \frac{p^2}{w}\right)}{3p} = \frac{2\left(\frac{1}{\sqrt{12}}24 + \frac{1}{\frac{1}{\sqrt{12}}}\right)}{3} = \frac{2\left(2\sqrt{12} + \sqrt{12}\right)}{3} = 2\sqrt{12}.$$

Thus, we have

$$z(w^*, p^*) + x_1(w^*, p^*) = 12 + 12 = 24 = \bar{L}$$
$$\text{and} \quad x_2(w^*, p^*) = 2\sqrt{12} = x_2(w^*, p^*).$$

The equilibrium is illustrated in figure 3.23. □

3.3 2 × 2 Production Model

In this model, we study the general equilibrium effects in production. To be more specific, we are interested in the general equilibrium of the factor (input) markets. To that end, we consider an economy with two input goods and two output goods where input prices are endogenously determined while output prices are exogenously fixed. An example we have in mind is a small open economy that trades its output goods in the world market but whose factors of production are domestic (not mobile across countries). The specification of the economy as follows.

- $L = 4$. Two goods are used as factors of (or, inputs to) production and labeled factor ℓ (or, input good ℓ), $\ell = 1, 2$. The remaining two goods are produced as outputs and labeled output good j, $j = 1, 2$.
- Let $w = (w_1, w_2) \gg 0$ be the input prices and let (p_1, p_2) be the output prices. Assume that $(p_1, p_2) \gg 0$ is exogenously fixed.
- Individuals are unmodeled and exist only to supply an aggregate endowment $\bar{\omega} = (\bar{z}_1, \bar{z}_2)$, where $\bar{z}_\ell$ is the endowment of input good ℓ. Alternatively, we can let $I = \bar{I}$ with aggregate endowment $\bar{\omega} = (\bar{z}_1, \bar{z}_2)$ and assume that individuals have have utility function $u_i(x_{1i}, x_{2i})$, where x_{ji} is the amount of the output good j, that is increasing in x_{ji}. Then the individuals will collectively supply $(\bar{z}_1, \bar{z}_2)$ at all (positive) prices.
- $J = 2$. Firm j uses the two inputs to produce output good j using production function $f_j(z_{1j}, z_{2j})$. Assume that the production technology is constant returns to scale (CRS). That is, $f_j(\alpha z_{1j}, \alpha z_{2j}) = \alpha f_j(z_{1j}, z_{2j})$ for all $\alpha > 0$.

As usual, firms maximize profit. Firm j's profit maximization problem can be written as

$$\max_{z_{1j}, z_{2j}} \; p_j f_j(z_{1j}, z_{2j}) - (w_1 z_{1j} + w_2 z_{2j}) \,. \tag{PMP}$$

The solution,

$$z_j(w_1, w_2) = (z_{1j}(w_1, w_2), z_{2j}(w_1, w_2)) \,,$$

is called the *(unconditional) input (factor) demand*, and

$$y_j(w_1, w_2) = f(z_{1j}(w_1, w_2), z_{2j}(w_1, w_2))$$

is called the *supply correspondence*.

An alternative formulation of profit maximization involves first deriving the cost function. Firm j's cost minimization problem is

$$\min_{z_{1j}, z_{2j}} \; w_1 z_{1j} + w_2 z_{2j} \quad \text{s.t.} \quad f_j(z_{1j}, z_{2j}) \geq q. \tag{CMP}$$

The solution,

$$z_j(w_1, w_2, q) = (z_{1j}(w_1, w_2, q), z_{2j}(w_1, w_2, q)) \,,$$

is called the *conditional input (factor) demand*. The unconditional input demand is the profit maximizing input level. In contrast, conditional input demand is cost minimizing input level, conditioned on producing some (not necessarily profit-maximizing) output level q. The value function of the cost minimization problem,

$$c_j(w_1, w_2, q) = w_1 z_{1j}(w_1, w_2, q) + w_2 z_{2j}(w_1, w_2, q).$$

is called the *cost function*. Once the cost function has been found, profit maximization problem can be formulated as

$$\max_q p\, q - c_j(w_1, w_2, q). \tag{PMP}$$

As before, the solution to this problem, $y_j(w_1, w_2)$ is firm j's supply correspondence.

The firm's production function can be visualized using isoquants, which are graphs of all the input combinations that produce the same output level, as in figure 3.24. For the remainder of the 2×2 production model, we will assume that the production functions are strongly monotone with strictly convex isoquants, which implies that the cost minimization problem has a unique solution. In particular, as illustrated in the figure, the solution occurs where the isoquant corresponding to the required output level q is tangent to an isocost line.

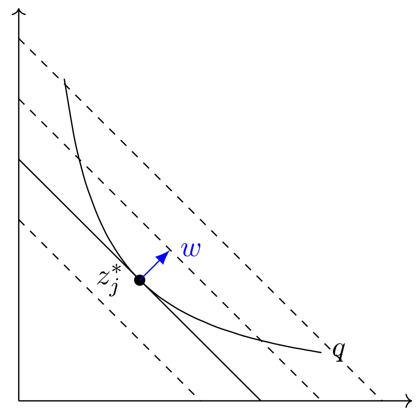

Figure 3.24: Cost minimization.

For differentiable production functions, the interior solution $z_j^* = (z_{1j}^*, z_{2j}^*)$ can be characterized by the usual first order conditions:

$$MRTS_j = \left.\frac{\frac{\partial f_j}{\partial z_{1j}}}{\frac{\partial f_j}{\partial z_{2j}}}\right|_{(z_{1j}^*, z_{2j}^*)} = \frac{w_1}{w_2} \qquad \text{(tangency)}$$

$$\text{and} \quad f_j\big(z_{1j}^*, z_{2j}^*\big) = q. \qquad \text{(output equation)}$$

The first condition states that the solution occurs where slope of the isoquants $(-MRTS_j)$ and the slope of the isocost line $(-\frac{w_1}{w_2})$ are equal, and the second condition further restricts the solution to be on the isoquant corresponding to the output level q.

Example 3.16. Suppose firm j's has a Cobb-Douglas production function:

$$f_j(z_{1j}, z_{2j}) = Az_{1j}^a z_{2j}^b, \quad A > 0 \text{ and } a, b > 0.$$

Its cost minimization problem is

$$\min_{z_{1j}, z_{2j}} \; w_1 z_{1j} + w_2 z_{2j} \quad \text{s.t.} \quad Az_{1j}^a z_{2j}^b \geq q.$$

To find the conditional input demand, we use the "MRTS = price ratio" condition to obtain

$$MRTS_j = \frac{\frac{\partial f_j}{\partial z_{1j}}}{\frac{\partial f_j}{\partial z_{2j}}} = \frac{Aaz_{1j}^{a-1} z_{2j}^b}{Abz_{1j}^a z_{2j}^{b-1}} = \frac{az_{2j}}{bz_{1j}} = \frac{w_1}{w_2} \implies z_{2j} = \left(\frac{bw_1}{aw_2}\right) z_{1j}.$$

Substitute this into the output equation to obtain

$$Az_{1j}^a \left(\left(\frac{bw_1}{aw_2}\right) z_{1j}\right)^b = q \implies z_{1j}^{a+b} = \left(\frac{aw_2}{bw_1}\right)^b \frac{q}{A}$$

$$\implies z_{1j}(w_1, w_2, q) = \left(\frac{aw_2}{bw_1}\right)^{\frac{b}{a+b}} \left(\frac{q}{A}\right)^{\frac{1}{a+b}}.$$

$$\implies z_{2j}(w_1, w_2, q) = \left(\frac{bw_1}{aw_2}\right) \left(\frac{aw_2}{bw_1}\right)^{\frac{b}{a+b}} \left(\frac{q}{A}\right)^{\frac{1}{a+b}} = \left(\frac{bw_1}{aw_2}\right)^{\frac{a}{a+b}} \left(\frac{q}{A}\right)^{\frac{1}{a+b}}.$$

The cost function is

$$\begin{aligned}
c_j(w_1, w_2, q) &= w_1 z_{1j}(w_1, w_2, q) + w_2 z_{2j}(w_1, w_2, q) \\
&= w_1 \left(\frac{aw_2}{bw_1}\right)^{\frac{b}{a+b}} \left(\frac{q}{A}\right)^{\frac{1}{a+b}} + w_2 \left(\frac{bw_1}{aw_2}\right)^{\frac{a}{a+b}} \left(\frac{q}{A}\right)^{\frac{1}{a+b}} \\
&= \left[\left(\frac{a^b}{b^b} w_1^a w_2^b\right)^{\frac{1}{a+b}} + \left(\frac{b^a}{a^a} w_1^a w_2^b\right)^{\frac{1}{a+b}}\right] \left(\frac{q}{A}\right)^{\frac{1}{a+b}} \\
&= \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}}\right] w_1^{\frac{a}{a+b}} w_2^{\frac{b}{a+b}} \left(\frac{q}{A}\right)^{\frac{1}{a+b}}
\end{aligned}$$

Specializing to the constant returns to scale means $a + b = 1$. If in addition we assume that $A = 1$, the conditional input demands and the cost function reduce to

$$\begin{aligned}
z_{1j}(w_1, w_2, q) &= \left(\frac{aw_2}{bw_1}\right)^b q \\
z_{2j}(w_1, w_2, q) &= \left(\frac{bw_1}{aw_2}\right)^a q \\
c_j(w_1, w_2, q) &= \left[\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a\right] w_1^a w_2^b q.
\end{aligned}$$

□

As the previous example shows, the conditional input demands and the cost function are linear in output quantity, q, for constant returns to scale Cobb-Douglas production functions. This hold generally for constant returns to scale production functions. That is,

$$c_j(w_1, w_2, q) = q c_j(w_1, w_2, 1).$$

This means that the firm cannot make positive profit in equilibrium. Otherwise, the firm can always scale up its production and make even larger profit:

$$pq - c_j(w, q) > 0 \implies p\alpha q - c_j(w, \alpha q) = \alpha\big(pq - c_j(w, q)\big) > pq - c_j(w, q)$$

for any $\alpha > 1$. Therefore, the firm must be making zero profit in equilibrium, and any multiple of $z_j(w_1, w_2, 1) = (z_{1j}(w_1, w_2, 1), z_{2j}(w_1, w_2, 1))$ is a solution to the profit maximization problem.

Before, proceeding, we give the relationship between the cost function and the conditional input known as *Shepard's Lemma*, which we state without proof (it arises from the envelope theorem).

$$\boxed{\frac{\partial c_j(w_1, w_2, q)}{\partial w_\ell} = z_{\ell j}(w_1, w_2, q)} \qquad \text{(Shepard's Lemma)}$$

Example 3.17. Taking the partial derivative of the cost function in 3.16 with respect to w_1 yields the conditional input demand for good 1:

$$\begin{aligned}\frac{\partial c_j(w_1,w_2,q)}{\partial w_1} &= \frac{\partial}{\partial w_1}\left(\left[\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a\right] w_1^a w_2^b q\right)\\ &= \left[\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a\right] a w_1^{a-1} w_2^b q = \left[\frac{aa^b}{b^b} + \frac{a^b b^a}{1}\right]\left(\frac{w_2}{w_1}\right)^b q\\ &= \left[\frac{aa^b + a^b b}{b^b}\right]\left(\frac{w_2}{w_1}\right)^b q = \left(\frac{a}{b}\right)^b \left(\frac{w_2}{w_1}\right)^b q = z_1(w_1,w_2,q).\end{aligned}$$

□

Graphically, this means that the the gradient of the cost function is the conditional input demand. Figure 3.25 illustrates this for unit cost cost curve, which is the cost function for producing one unit of output. It also means that the magnitude of the slope of the cost function is the factor intensity ratio. For example, along the unit isocost curve,

$$\frac{dw_2}{dw_1} = -\frac{\frac{\partial c_j(w_1,w_2,1)}{\partial w_1}}{\frac{\partial c_j(w_1,w_2,1)}{\partial w_2}} = -\frac{z_{1j}(w_1,w_2,1)}{z_{2j}(w_1,w_2,1)}.$$

Because cost functions are concave in input prices, isocost curves are convex (exercise). This implies that the factor intensity ratio is decreasing as we move along the curve in the direction of increasing w_1 and decreasing w_2.

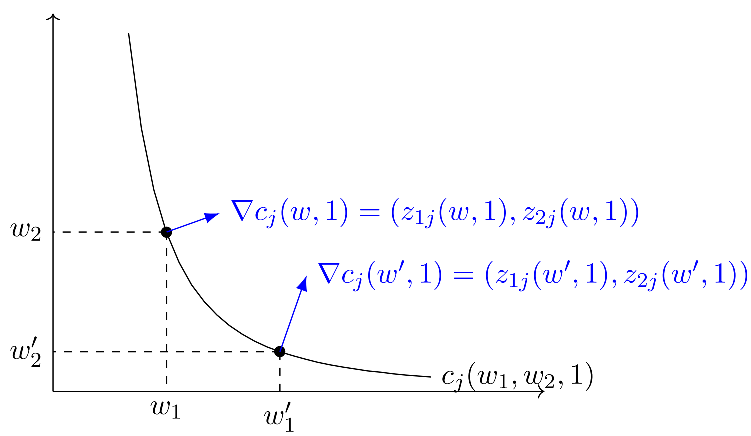

Figure 3.25: Unit cost function.

The notion of efficiency in the factor market is Pareto optimality, and it is analogous to Pareto optimality in an Edgeworth box economy, with isoquants replacing the role of indifference curves. In particular, for differentiable production functions, $(z_1,z_2) = \big((z_{11},z_{21}),(z_{12},z_{22})\big)$ is an interior Pareto optimal allocation if

$$\text{MRTS}_1\big|_{(z_{11},z_{21})} = \text{MRTS}_2\big|_{(z_{12},z_{22})} \quad \text{and} \quad (z_{11},z_{21}) + (z_{12},z_{22}) = (\bar{z}_1,\bar{z}_2).$$

For the previous example, suppose the total factor endowment is $(10, 6)$.

$$\text{MRTS}_1 = \frac{a_1 z_{21}}{b_1 z_{11}} = \frac{\frac{2}{3} z_{21}}{\frac{1}{3} z_{11}} = \frac{2z_{21}}{z_{11}} = \frac{z_{22}}{2z_{12}} = \frac{\frac{1}{3} z_{22}}{\frac{2}{3} z_{12}} = \frac{a_2 z_{22}}{b_2 z_{12}} = \text{MRTS}_2$$

$$\implies \frac{2z_{21}}{z_{11}} = \frac{6 - z_{21}}{2(10 - z_{11})}$$

$$40z_{21} - 4z_{11}z_{21} = 6z_{11} - z_{11}z_{21}$$

$$z_{21} = \frac{6z_{11}}{40 - 3z_{11}}.$$

The Pareto set is illustrated in figure 3.26 as the thick blue line.

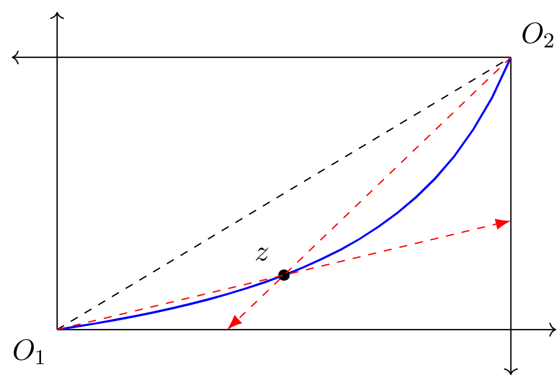

Figure 3.26: Pareto set for factor market.

Taking the outputs of the two firms from each Pareto optimal input allocation and plotting them in the q_1-q_2 axis produces the *production possibility set* (assuming free disposal). The boundary of the production possibility set is the production possibility frontier (see figure 3.27).

Figure 3.27: Production possibility set.

The Pareto set in figure 3.26 lies on one side of the diagonal line. It turns out that when the production functions are constant returns to scale (or, more generally, *homothetic*), the Pareto set either lies entirely on one side of the diagonal line (except the corner points) or coincides with the diagonal. Moreover, when the Pareto set is not the diagonal line, the factor intensity $\frac{z_{1j}}{z_{2j}}$ of one firm is greater than that of the other at every point along the Pareto set (except the corner points). For example, in figure 3.26,

$$\frac{z_{11}}{z_{21}} > \frac{\bar{z}_1}{\bar{z}_2} > \frac{z_{12}}{z_{22}}, \left(\text{or equivalently, } \frac{z_{21}}{z_{11}} < \frac{\bar{z}_2}{\bar{z}_1} < \frac{z_{22}}{z_{12}}\right).$$

Moreover, the factor intensities change monotonically along the Pareto set. For example, in figure 3.26, factor intensity of both firms decrease from O_1 to O_2.

To examine the nature of the factor market equilibrium more closely, we will assume that the factor intensity of one firm is always larger than the other firm.

Assumption 3.18 (*Factor intensity assumption*)**.** Production of good 1 is relatively more intensive in factor 1 than the production of good 2. That is,

$$\frac{z_{11}(w_1, w_2, 1)}{z_{21}(w_1, w_2, 1)} > \frac{z_{12}(w_1, w_2, 1)}{z_{22}(w_1, w_2, 1)}$$

at all factor prices (w_1, w_2). □

Example 3.19. Suppose the two firms' production functions are

$$f_1(z_{11}, z_{21}) = z_{11}^{\frac{2}{3}} z_{21}^{\frac{1}{3}} \quad \text{and} \quad f_2(z_{12}, z_{22}) = z_{12}^{\frac{1}{3}} z_{22}^{\frac{2}{3}}.$$

Then the production of good 1 is relatively more intense than the production of good 2. To see this, first note that or a generic CRS Cobb-Douglas production function,

$$f_j(z_{1j}, z_{2j}) = z_{1j}^{a_j} z_{2j}^{b_j},$$

we have

$$\frac{z_{1j}(w_1, w_2, 1)}{z_{2j}(w_1, w_2, 1)} = \frac{\left(\frac{a_j w_2}{b_j w_1}\right)^{b_j}}{\left(\frac{b_j w_1}{a_j w_2}\right)^{a_j}} = \frac{a_j w_2}{b_j w_1}.$$

Therefore,

$$\frac{z_{11}(w_1, w_2, 1)}{z_{21}(w_1, w_2, 1)} = \frac{\frac{2}{3} w_2}{\frac{1}{3} w_1} = \frac{2 w_2}{w_1} > \frac{w_2}{2 w_1} = \frac{\frac{1}{3} w_2}{\frac{2}{3} w_1} = \frac{z_{12}(w_1, w_2, 1)}{z_{22}(w_1, w_2, 1)}.$$

□

We look for an equilibrium in which both goods are being produced in the economy. Then for $w^* = (w_1^*, w_2^*)$ to be an equilibrium factor prices, the firms must be making zero profit in equilibrium. Otherwise, constant returns to scale means firms' either want to produce infinite amount of good or shut down. Thus, w_1^* and w_2^* are found by solving the two zero-profit conditions.

$$\boxed{p_1 = c_1(w_1^*, w_2^*, 1) \quad \text{and} \quad p_2 = c_2(w_1^*, w_2^*, 1).}$$

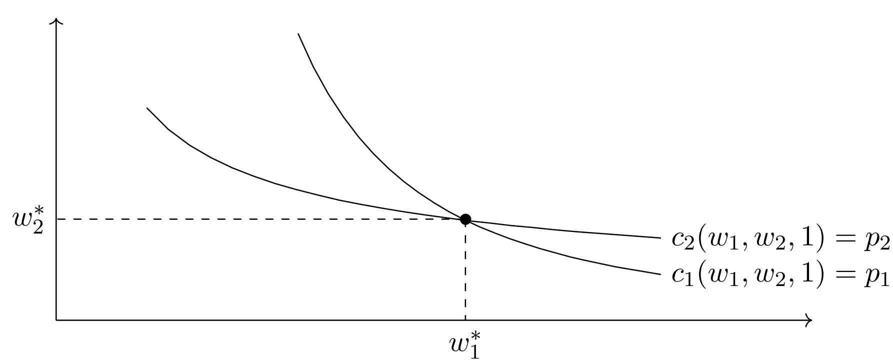

Figure 3.28: Candidate for equilibrium factor prices.

Since the firms are making zero profit, any scaling of a conditional input demand is also profit maximizing. Thus, to find the equilibrium factor allocation, we first find the unit conditional input demands at w^* and then scale them to satisfy the market clearing conditions, as in figure 3.29 (the scale factor is also the output level). As the figure shows, the existence of an interior equilibrium allocation (the intersection of the two rays) requires the aggregate factor endowments $(\bar{z}_1, \bar{z}_2)$ to satisfy the following additional condition.

$$\boxed{\frac{z_{11}(w_1^*, w_2^*, 1)}{z_{21}(w_1^*, w_2^*, 1)} > \frac{\bar{z}_1}{\bar{z}_2} > \frac{z_{12}(w_1^*, w_2^*, 1)}{z_{22}(w_1^*, w_2^*, 1)}.}$$

Example 3.20. Consider a factor market with productions given by:

$$f_1(z_{11}, z_{21}) = z_{11}^{\frac{2}{3}} z_{21}^{\frac{1}{3}} \quad \text{and} \quad f_2(z_{12}, z_{22}) = z_{12}^{\frac{1}{3}} z_{22}^{\frac{2}{3}}.$$

Let $p_1 = 16$ and $p_2 = 10$. To find the factor market equilibrium prices, we solve the two zero-profit conditions. In example 3.16, the unit cost function of firm j was found to be

$$c_j(w_1, w_2, 1) = \left[\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a\right] w_1^a w_2^b.$$

Thus, the zero profit condition for firm 1 yields

$$16 = \left[\left(\frac{\frac{2}{3}}{\frac{1}{3}}\right)^{\frac{1}{3}} + \left(\frac{\frac{1}{3}}{\frac{2}{3}}\right)^{\frac{2}{3}}\right] w_1^{\frac{2}{3}} w_2^{\frac{1}{3}} = \left[2^{\frac{1}{3}} + \left(\frac{1}{2}\right)^{\frac{2}{3}}\right] w_1^{\frac{2}{3}} w_2^{\frac{1}{3}} = 1.890\, w_1^{\frac{2}{3}} w_2^{\frac{1}{3}}$$

$$\implies w_1^2 w_2 = \left(\frac{16}{1.890}\right)^3 = 606.70 \implies w_2 = \frac{606.70}{w_1^2}.$$

The zero profit condition for firm 2 yields

$$10 = \left[\left(\frac{1}{2}\right)^{\frac{2}{3}} + 2^{\frac{1}{3}} +\right] w_1^{\frac{1}{3}} w_2^{\frac{2}{3}} = 1.890 w_1^{\frac{1}{3}} w_2^{\frac{2}{3}}$$

$$\implies w_1 w_2^2 = \left(\frac{10}{1.890}\right)^3 = 148.12 \implies w_1 \left(\frac{606.70}{w_1^2}\right)^2 = 148.12$$

$$w_1^* = \left(\frac{606.70^2}{148.12}\right)^{\frac{1}{3}} = 13.54$$

$$w_2^* = \frac{606.70}{13.54^2} = 3.31.$$

To find the equilibrium factor allocation, we will first find the unit conditional input demand. In example 3.16, conditional input demand of firm j was found to be:

$$z_{1j}(w_1^*, w_2^*, 1) = \left(\frac{a w_2}{b w_1}\right)^b q$$

$$z_{2j}(w_1^*, w_2^*, 1) = \left(\frac{b w_1}{a w_2}\right)^a q.$$

Therefore, we have

$$z_{11}(w_1^*, w_2^*, 1) = \left(\frac{\frac{2}{3}(3.31)}{\frac{1}{3}(13.54)}\right)^{\frac{1}{3}} = \left(\frac{2(3.31)}{13.54}\right)^{\frac{1}{3}} = 0.788$$

$$z_{21}(w_1^*, w_2^*, 1) = \left(\frac{\frac{1}{3}(13.54)}{\frac{2}{3}(3.31)}\right)^{\frac{2}{3}} = \left(\frac{13.54}{2(3.31)}\right)^{\frac{2}{3}} = 1.611.$$

$$z_{12}(w_1^*, w_2^*, 1) = \left(\frac{\frac{1}{3}(3.31)}{\frac{2}{3}(13.54)}\right)^{\frac{2}{3}} = \left(\frac{3.31}{2(13.54)}\right)^{\frac{2}{3}} = 0.246.$$

$$z_{22}(w_1^*, w_2, 1) = \left(\frac{\frac{2}{3}(13.54)}{\frac{1}{3}(3.31)}\right)^{\frac{1}{3}} = \left(\frac{2(13.54)}{3.31}\right)^{\frac{1}{3}} = 2.014.$$

Now suppose, the aggregate factor endowments are $\bar{z}_1 = 6$ and $\bar{z}_2 = 20$. These endowments were chosen so that $\frac{\bar{z}_1}{\bar{z}_2} = 0.3$ satisfies the requirement on the aggregate endowment ratio mentioned earlier:

$$\frac{z_{11}(w_1^*, w_2^*, 1)}{z_{21}(w_1^*, w_2^*, 1)} = \frac{0.788}{1.611} = 0.489 > 0.3 > 0.122 = \frac{0.246}{2.014} = \frac{z_{12}(w_1^*, w_2^*, 1)}{z_{22}(w_1^*, w_2^*, 1)}.$$

Now, we find the scaling required to clear the factor markets. That is we find, $\alpha > 0$ and $\beta > 0$ so that $\alpha z(w_1^*, w_2^*, 1) + \beta z(w_1^*, w_2^*, 1) = \bar{z}$. We write this as a column vector equation to make the scaling clearer.

$$\alpha \begin{bmatrix} 0.788 \\ 1.611 \end{bmatrix} + \beta \begin{bmatrix} 0.246 \\ 2.014 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \end{bmatrix}.$$

The first equation yields

$$\alpha = \frac{6 - 0.246\beta}{0.788} = 7.614 - 0.312\beta.$$

Substitute this into the second equation to obtain

$$\begin{aligned}(7.614 - 0.312\beta)(1.611) = 2.014\beta = 20 \\ \implies \beta = \frac{7.734}{1.511} = 5.118 \\ \alpha = 7.614 - 0.312(5.118) = 6.017.\end{aligned}$$

Therefore, the equilibrum output levels are $q_1^* = \alpha = 6.017$ and $q_2^* = \beta = 5.118$. And the equilibrium factor allocations are:

$$\begin{aligned}z_{11}(w_1^*, w_2^*, q_1^*) &= 6.017(0.788) = 4.741 \\ z_{21}(w_1^*, w_2^*, q_1^*) &= 6.017(1.611) = 9.693 \\ z_{12}(w_1^*, w_2^*, q_1^*) &= 5.118(0.246) = 1.259 \\ z_{22}(w_1^*, w_2^*, q_1^*) &= 5.118(2.014) = 10.398.\end{aligned}$$

To verify that the markets indeed clear, note that $z_{11}^* + z_{12}^* = 4.741 + 1.259 = 6 = \bar{z}_1$ and $z_{21}^* + z_{22}^* = 9.693 + 10.398 = 20.091 \approx 20 = \bar{z}_2$. The discrepancy is the rounding error. The equilibrium is illustrated in figure 3.29. □

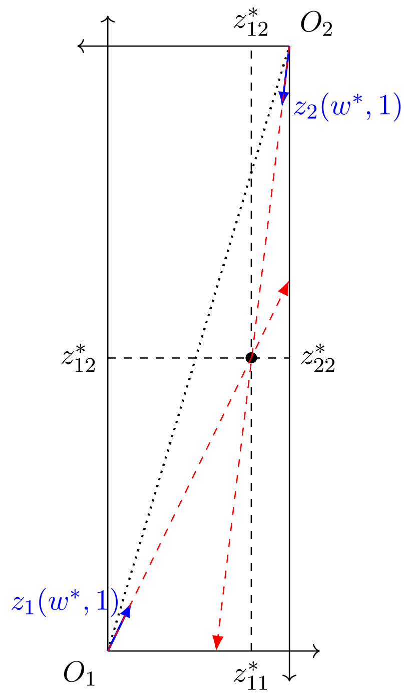

Figure 3.29: Factor market equilibrium.

In Example 3.20, the determination of the equilibrium factor prices did not depend on the details of the factor endowments. All that was relevant was that the ratio of the aggregate factor endowments fall somewhere between the two firms' factor intensities, and even that was to obtain the interior equilibrium allocation, where both firms produce. Thus, assuming that both goods are produced, this implies that if countries have identical production technologies and act as a price-taker in the output goods, which are traded in the world market, then they will have the same factor prices even though the factors are not traded internationally. This is result is called the *factor price equalization theorem* in the international trade literature.

In the remainder of this subsection we study how the economy changes when underlying parameters of the economy change. Because transition dynamics are not well understood, our analysis is limited to comparative statics, which compares two static situations. In particular, we compare the original equilibrium with the new equilibrium after a parameter changes and all the adjustment process has been completed. We do not study the dynamic process by which the economy transitions from one equilibrium to another.

The first comparative statics result shows what happens to the economy when one of the output prices, which are assumed to be exogenous, changes. For example, if p_2 increases to p_2', then, the unit cost curve corresponding the the zero-profit condition for firm 2 shifts out since higher input prices can support the higher output price, as illustrated in figure 3.30. As the figure shows, the new intersection, meaning the new equilibrium factor prices $w' = (w_1', w_2')$, moves higher and to the left of the original intersection. Thus, the equilibrium price of factor 2 increases and that of factor 1 decreases. This result is known as *Stolper-Samuelson Theorem*. Since the gradient of c_1 is flatter at w' than w^* and the gradient of c_2 is also flatter at w' than w^*, the equilibrium allocation of the factor moves toward O_1. Thus, an implication of Stolper-Samuelson Theorem is that the greater amount of both factors are allocated to the firm whose output price increased, which also means that the output of that firm increases at the expense of the other firm.

Theorem 3.21 (Stolper-Samuelson Theorem)**.** *In the 2×2 production model with the factor intensity assumption, suppose the price of output good j increases and the economy produces both goods both before and after the price increase. Then the equilibrium price of the factor more intensively used in the production of good j increases, while the price of the other factor decreases.*

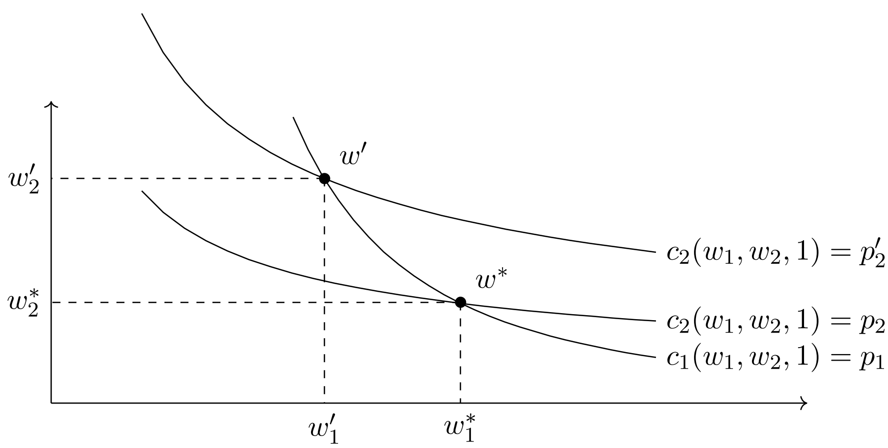

Figure 3.30: Changes in the equilibrium factor prices when $p_2 \uparrow p'_2$.

The second comparative statics result studies what happens when the total endowment of one of the factors increase. Suppose, for example, the endowment of factor 1 increases to $\bar{z}'_1$ and the economy produces both goods both before and after the endowment increase. Then because output prices did not change, the equilibrium input prices do not change either. Given the economy still has the same production function, this means that the equilibrium factor intensity does not change either. Thus, new equilibrium can be found by increasing the length of the factor Edgeworth box, as in figure 3.31. In the figure, the blue parts are the additions arising from the increase in the endowment of factor 1. The equilibrium factor allocation moves to the intersection of the blue dotted rays, which means more of both factors are allocated to firm 1 and fewer of both are allocated to firm 2, which means that the output of firm 1 increases at the expense of firm 2. This result is known as *Rybczynski Theorem*.

Theorem 3.22 (Rybczynski Theorem)**.** *In the 2×2 production model with the factor intensity assumption, suppose the endowment of a factor increases and the economy produces both goods both before and after the price increase. Then the production of the good that uses this factor more intensively increases and* *the production of the other good decreases.*

Note that the figure also suggests that if the endowment of factor 1 increases too much than the new rays will not intersect and the interior equilibrium will not exist. This is because factor intensity assumption would then be violated.

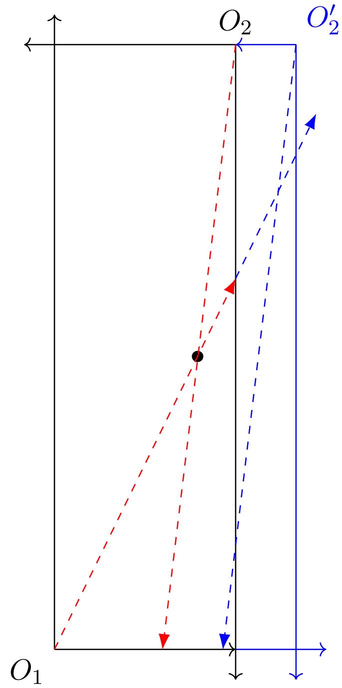

Figure 3.31: Changes in the equilibrium when $\bar{z}_1 \uparrow \bar{z}_1'$.