ADVANCED MICRO I.
SWUFE
2010 FALL

I. CLASSICAL DEMAND THEORY

- PREFERENCE BASED APPROACH – ASSUME CONSUMERS HAVE WELL-DEFINED PREFERENCE ORDERING OVER THE CHOICE SETS AND CHOOSE THE ONE THEY LIKE THE BEST.
- STARTED AS AN ALTERNATIVE TO UTILITY BASED APPROACH
- SET UP

LET $X \equiv \mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n : x \geq 0\}$ ($\subseteq \mathbb{R}^n$ (SET TYPICALLY))
BE THE CONSUMPTION SET.

LET $x \in X \equiv (x_1, x_2, \ldots, x_n) \equiv$ COMMODITY BUNDLE

$x_i \equiv$ AMT OF i-th GOOD.

LET $\succsim \equiv$ PREFERENCE ORDERING (RELATION)

$x \succsim y \equiv x$ IS "WEAKLY PREFERRED" TO y
$\equiv x$ IS "AT LEAST AS GOOD AS" y

$x \sim y \equiv$ "INDIFFERENT" BETWEEN x & y
$\equiv x$ IS "LIKED EQUALLY AS MUCH" AS y

$x \succ y \equiv x$ IS "STRICTLY PREFERRED" TO y
$\equiv x$ IS "BETTER THAN" y
$\equiv x \succsim y$ BUT NOT $x \sim y$

NOTE. $\geq$ IS A MATH RELATION COMPARING QUANTITY
$\succsim$ IS AN ECONOMIC RELATION COMPARING SATISFACTION LEVEL

- <u>DEFN</u>: PREF. ORDERING $\succsim$ IS <u>RATIONAL</u> IF IT SATISFIES

① <u>COMPLETENESS</u>: $\forall x, y \in X$, HAVE $x \succsim y$ OR $y \succsim x$.

I.E. ANY TWO BUNDLES IN THE CONSUMPTION SET CAN BE ORDERED.

② <u>TRANSITIVITY</u>: $\forall x, y, z \in X$, IF $x \succsim y$ AND $y \succsim z$, THEN $x \succsim z$.

I.E. IF x IS AT LEAST AS GOOD AS y & y IS AS GOOD AS z THEN x IS AT LEAST AS GOOD AS z.

- <u>DEFN</u>: PREF ORDERING $\succsim$ IS <u>CONTINUOUS</u> IF $\forall$ SEQUENCES $\{x_n : n \geq 1\}$ AND $\{y_n : n \geq 1\}$ S.T. $x_n \succsim y_n$ $\forall n$, $x_n \to x$, AND $y_n \to y$, WE HAVE $x \succsim y$.

I.E., $\succsim$ IS PRESERVED UNDER THE LIMIT OPERATOR.

	APPLE	BANANA		APPLE	BANANA	
E.G. $x_1 =$	(2.1 ,	3.1)	$\succsim$	(4.1 ,	2.1)	$= y_1$
$x_2 =$	(2.01 ,	3.01)	$\succsim$	(4.01 ,	2.01)	$= y_2$
$x_3 =$	(2.001 ,	3.001)	$\succsim$	(4.001 ,	2.001)	$= y_3$
	⋮				⋮	
	x_n		$\succsim$		y_n	
	⋮				⋮	

$x_n \to x = (2,3)$ $\qquad$ $y_n \to y = (4,2)$

THEN $(2,3) \succsim (4,2)$ IF $\succsim$ IS CONTINUOUS.

I.E., PREF. ORDERING DOESN'T SUDDENLY REVERSE IN THE LIMIT.

- RATIONALITY (① & ②) IS A KEY ASSUMPTION
CONTINUITY IS A TECHNICAL ASSUMPTION.
UNLESS STATED O.W., WE'LL ALWAYS ASSUME ①, ②, ③

(DO MONOTONICITY FIRST)

- DEFN: $\succsim$ IS LOCALLY-NONSATIATED IF

$\forall x \in X$ & $\varepsilon > 0$, $\exists y \in X$ s.t. $\|y-x\| \leq \varepsilon$

AND $y \succ x$.

I.E., THERE IS NO COMPLETELY SATISFYING BUNDLE: FOR ANY $x \in X$ THERE IS SOME CLOSE BY BUNDLE THAT THE CONSUMER STRICTLY PREFERS

- DEFN: ① $\succsim$ IS (WEAKLY) MONOTONE IF

$x \gg y \Rightarrow x \succ y$.

② $\succsim$ IS STRONGLY MONOTONE IF

$x > y \Rightarrow x \succ y$

MONOTONICITY IS A "MORE IS BETTER" PROPERTY

IF $\succsim$ IS WEAKLY MONOTONE, THEN WHENEVER y HAS MORE OF EVERY GOOD THAN x, y IS STRICTLY PREFERRED TO x.

IF $\succsim$ IS STRONGLY MONOTONE, THEN WHENEVER y HAS AS MUCH OF EVERY GOOD THAN x AND ~~HAS~~ HAS MORE THAN x OF SOME GOOD, THEN y IS PREFERRED TO x.

E.G. $\succsim$ IS STRONGLY MONOTONE $\Rightarrow (3,2) \succ (2,2)$

BUT $\succsim$ IS WEAKLY MONOTONE $\Rightarrow (3,2) \; ? \; (2,2)$

DEFN: GIVEN $\succsim$, AN UPPER CONTOUR SET OF x IS $\{y \in X : y \succsim x\}$ $\equiv$ THE SET OF ALL BUNDLES THAT ARE AT LEAST AS GOOD AS x.

DEFN: ① $\succsim$ IS (WEAKLY) CONVEX IF $\forall y \succsim x$, $z \succsim x$ AND $\alpha \in [0,1]$, $\alpha y + (1-\alpha) z \succsim x$.

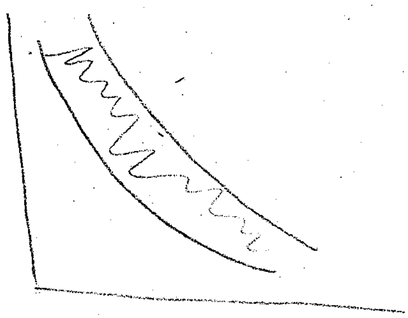
② $\succsim$ IS STRICTLY CONVEX IF $\forall y \succsim x$, $z \succsim x$ ST. $y \neq z$ AND $\alpha \in (0,1)$, HAVE $\alpha y + (1-\alpha) z \succ x$.

- I.E., $\succsim$ IS CONVEX IF THE UPPER CONTOUR SETS ARE CONVEX AND STRICTLY CONVEX IF THEY ARE STRICTLY CONVEX.

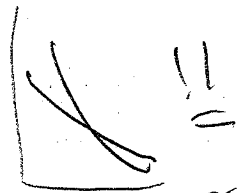
- ALTERNATIVELY, $\succsim$ IS CONVEX IF EVERY CONVEX COMB OF x & y IS AT LEAST AS GOOD AS THE WORSE OF THE TWO. SIMILARLY FOR STRICTLY CONVEX. ⟨S. PREFERRED TO⟩

- CONVEXITY CAPTURES "MODERATION IS BETTER THAN EXTREMES" IDEA

SOME FIGURES:

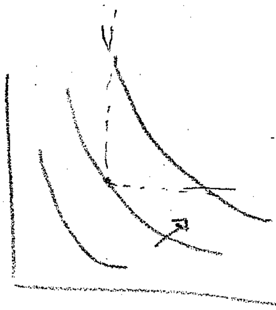
1) 

THIS INDIFF CURVES VIOLATES LOCAL NON-SATIATION

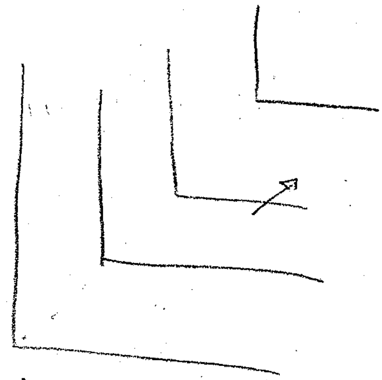
1B 

!!

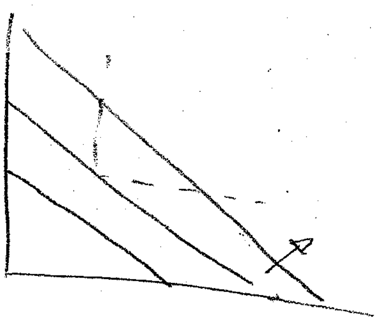
VIOLATES MONOTONICITY

2) 

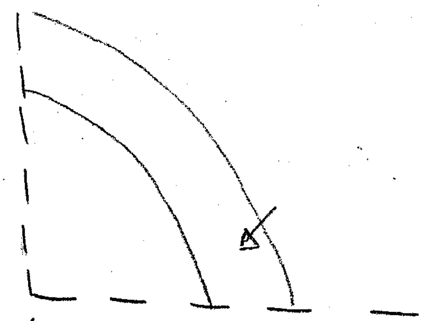
STRONGLY MONOTONE
STRICTLY CONVEX

3) 

WEAKLY MONOTONE
WEAKLY CONVEX

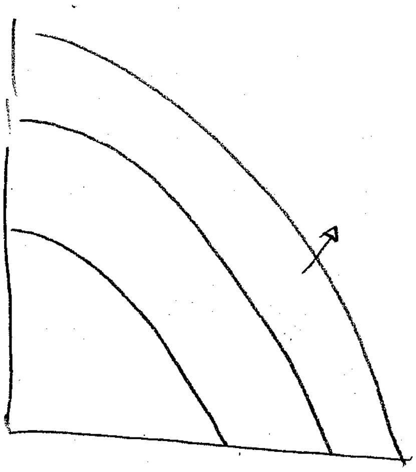
4) 

STRONGLY MONOTONE
WEAKLY CONVEX

5) 

$\mathbb{R}^n_{++}$

S. CONVEX
NOT MONOTONE
BUT LOCALLY NON-SATIATED

6) 

STRONGLY MONOTONE
"CONCAVE", NOT CONVEX

EXERCISE: $X = \mathbb{R}^n_+$

SHOW S. MONOTONICITY

$\Rightarrow$ W. MONOTONICITY

$\Rightarrow$ LOCAL NON-SATIATION

DEFN: FUNCTION $u: \mathbb{R}^n \to \mathbb{R}$ IS SAID TO BE A UTILITY FUNCTION REPRESENTING $\succsim$ IF

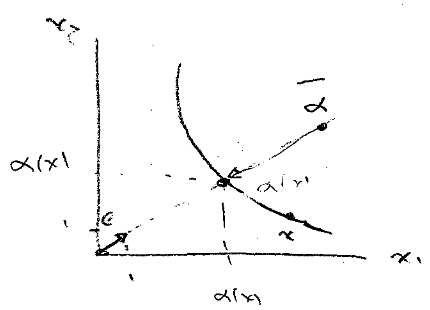
$$u(x) \geq u(y) \iff x \succsim y$$

- FOLLOWING THM CONNECTS "UTILITY BASED APPROACH TO CONSUMER CHOICE" TO "THE PREF. BASED APPROACH"

THM: SUPPOSE $\succsim$ IS A PREF ORDERING ON $X \subseteq \mathbb{R}^n$ THAT IS COMPLETE, TRANSITIVE, AND CONTINUOUS THEN THERE IS A CONTINUOUS UTILITY FUNCTION REPRESENTING $\succsim$.

SKETCH OF PF: RESTRICT TO $X = \mathbb{R}^2_+$ & $\succsim$ MONOTONE CASE.

LET $e = (1,1)$.
TAKE ANY $x \in \mathbb{R}^2_+$. LET $\bar{\alpha} e \gg x$.
SINCE $\bar{\alpha} e \gg x \geq 0$, WE HAVE
$\bar{\alpha} e \succ x \succsim 0$.
SO, AS WE DECREASE α FROM $\bar{\alpha}$ TO 0,
THERE MUST BE $\alpha(x)$ S.T. $\alpha(x) e \sim x$.
O.W. MONOTONICITY OR CONTINUITY IS VIOLATED



WE CAN SET $u(x) = \alpha(x)$. —//—

- NOTE NOTHING SPECIAL ABOUT α.
ANY INCREASING MONOT. TRANSF WILL DO.

- This is an important result. It allows us to frame any analysis involving $\succsim$ into that involving utility function.
- Note that desirability (local non-sat / monotonicity) or convexity assumptions are not needed for this result.

Consumer Choice Problems:

Assume $X = \mathbb{R}^n$

$\succsim$ is locally non satiated & add to (A)

Utility Maximization Problem

$$\max_{x \in \mathbb{R}^n_+} u(x) \quad \text{s.t.} \quad p \cdot x \le w \quad \left\langle \begin{matrix} p \gg 0 \\ w > 0 \end{matrix} \right\rangle \quad *$$

This is a problem of finding a bundle that yields the highest level of utility from those she can afford.

THM: Suppose $p \gg 0$ and $u(\cdot)$ is continuous. Then UMP has a solution.

- We'll assume $p \gg 0$ unless o.w. noted (*)
  - Solution to UMP depends on p & w & is called Marshallian / Walrasian demand.

If a solution is always single valued, it's called Marshallian demand function or simply demand function.

If a solution is multi-valued, called (Marsh) demand correspondence.

Denote it as $x(p, w)$.

$x_i(p, w)$ = demand for i-th good

THM: SUPPOSE $u(\cdot)$ IS A CONT. UTILITY FUNC. REPRESENTING LOCALLY NON-SATIATED $\succsim$. THE MARSHALLIAN DEMAND CORRESPONDENCE SATISFIES

① HOMOGENEITY OF DEGREE ZERO IN (p, w):

I.E. $\forall \alpha > 0$, $x(p, w) = x(\alpha p, \alpha w)$

② WALRAS' LAW: $p \cdot x(p, w) = w$

③ CONVEXITY / UNIQUENESS: • IF $\succsim$ IS CONVEX THEN, $x(p, w)$ IS A CONVEX SET.

• IF $\succsim$ IS STRICTLY CONVEX, THEN $x(p, w)$ IS A SINGLETON

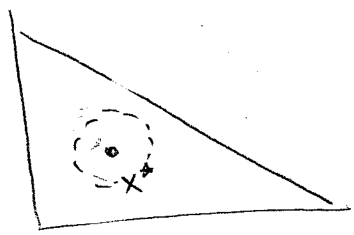
TO SEE: ① HD ϕ

$$\max_{x \in X} u(x) \quad \text{s.t. } (\alpha p) \cdot x \le \alpha w$$

$$\Leftrightarrow \max_{x \in X} u(x) \quad \text{s.t. } p \cdot x \le w$$

SAME PROBLEM SO, $x(\alpha p, \alpha w) = x(p, w)$

② W LAW:



SUPPOSE x^* IS STRICTLY INSIDE THE BUDGET SET THEN $\exists\, \varepsilon > 0$ S.T. $y \in X$ & $\|y - x^*\| < \varepsilon \Rightarrow$ y IS IN THE BUDGET.

BY LOCAL NON-SAT SATS GIVEN SUCH ε, $\exists$ y^* S.T. $y^* \succ x^*$ SO, x^* CANNOT BE OPTIMAL IF IT IS STRICTLY INSIDE THE BUDGET SET

(3) CONVEXITY: SUPPOSE x^* & y^* ARE TWO SOLUTIONS. (SO $x^* \sim y^*$) TAKE ANY $\alpha \in [0,1]$ &

LET $z^* = \alpha x^* + (1-\alpha) y^*$. NEED TO SHOW

z^* IS A SOLUTION TO U-MAX.

FIRST, z^* SATISFIES THE BUDGET CONSTRAINT

SINCE
$$p \cdot z^* = p \cdot (\alpha x^* + (1-\alpha) y^*)$$
$$= \alpha \underbrace{p \cdot x^*}_{\leq w} + (1-\alpha) \underbrace{p \cdot y^*}_{\leq w}$$
$$\leq w.$$

MOREOVER BY CONVEXITY, $z^* \succsim x^* \Rightarrow z^* \sim x^*$

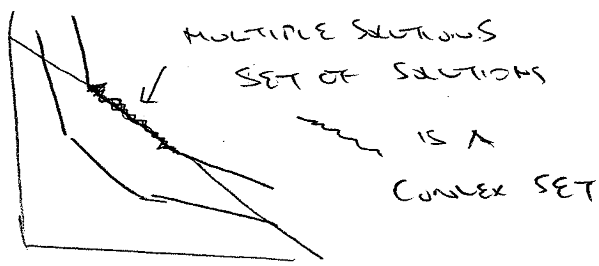
So, z^* IS A SOLUTION

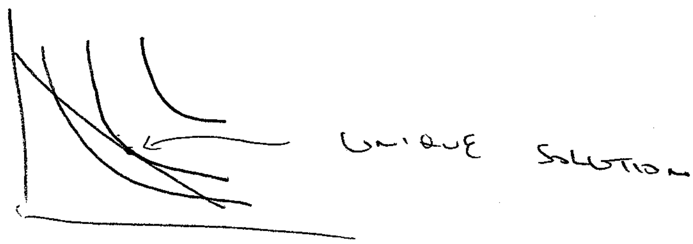
(4) IF $\succsim$ IS STRICTLY CONVEX, THEN

$z^* \succ x^*$ IF $x^* \neq y^*$. NOT POSSIBLE

So, $x^* = y^*$, MEANING ONLY ONE SOLUTION

PICTURE:





<DO NUMERICAL C-D FIRST !!!>

SOLVING UMP FOR DIFFERENTIABLE UTILITY FUNCTIONS, QUASI-CONCAVE UTILITY FUNC & INTERIOR SOLN.

HAVE $\max_{x \in \mathbb{R}^n_+} u(x)$ s.t. $p \cdot x \leq W$

WALRAS LAW $\Rightarrow$ $\max_{x \in \mathbb{R}^n_+} u(x)$ s.t. $p \cdot x = W$

$\Leftrightarrow \max_{x_1, x_2, \ldots, x_n \geq 0} u(x_1, x_2, \ldots, x_n)$ s.t. $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = W$

LAGRANGE MULTIPLIER METHOD <SEE APPENDIX>

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2, \ldots, x_n) + \lambda \left[W - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n \right]$$

FOC: $\frac{\partial \mathcal{L}}{\partial x_1} \leq 0 \Leftrightarrow \frac{\partial u}{\partial x_1} - \lambda p_1 \leq 0$ $\left\langle \begin{array}{l} = 0 \Leftarrow x_1^* > 0 \\ < 0 \Rightarrow x_1^* = 0 \end{array} \right\rangle$

$\frac{\partial \mathcal{L}}{\partial x_2} \leq 0 \Leftrightarrow \frac{\partial u}{\partial x_2} - \lambda p_2 \leq 0$

$\vdots$

$\frac{\partial \mathcal{L}}{\partial x_i} \leq 0 \Leftrightarrow \frac{\partial u}{\partial x_i} - \lambda p_i \leq 0$

$\vdots$

$\frac{\partial \mathcal{L}}{\partial x_n} \leq 0 \Leftrightarrow \frac{\partial u}{\partial x_n} - \lambda p_n \leq 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow W - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n = 0$

SOC: IF f IS QUASI-CONCAVE & GRADIENT OF f NEVER VANISHES <I.E. $\nabla f(x) \neq 0 \ \forall x$>

THEN NEC & SUFF. CONDITION FOR x^* TO ATTAIN GLOBAL MAX AT x^* IS THAT FOC ARE SATISFIED AT x^*.

SEE APPENDIX M.K.

(11)

I.e. ① $\frac{\partial u}{\partial x_i} \leq \lambda p_i \quad \forall i = 1, \dots, n$ $\Big|$ $=0$ IF $x_i > 0$; $\lambda x_i^* = 0$ IF ① < 0

② $p_1 x p_2 x \leq W$ B. EQUATION

① CAN BE WRITTEN AS: ⟨AT INT. SOLUTION⟩

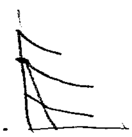
$$\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} = \frac{p_i}{p_j} \quad \forall i \,\&\, j \qquad MRS = \text{PRICE RATIOS}$$

HERE $\frac{\partial u/\partial x_1}{p_1} < \frac{\partial u/\partial x_2}{p_2}$ WANT TO BUY LESS GOOD 1 BUT $x_1 = 0$ ALREADY

NOTE: MRS ≠ PRICE RATIO AT BOUNDARY SOLN.

$x_1^* = 0, \; x_2^* \neq 0$

$\Rightarrow \frac{\partial u/\partial x_1}{\partial u/\partial x_2} \leq \frac{p_1}{p_2}$



① CAN ALSO BE WRITTEN AS

$$\frac{\frac{\partial u}{\partial x_i}}{p_i} = \frac{\frac{\partial u}{\partial x_j}}{p_j} \quad \forall i \,\&\, j$$

EXERCISE: INTERPRET:

SOC IGNORE IF ∇f NEVER VANISHES

NUMERICAL EXAMPLE: $u(x_1, x_2) = x_1^{1/3} x_2^{2/3}$ C-D UTILITY

$$\max_{x_1, x_2} x_1^{1/3} x_2^{2/3} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq W, \quad x_1 \geq 0, \quad x_2 \geq 0.$$

SINCE LOCALLY NON-SATIATED & SOLN WILL BE INTERIOR, REDUCE TO

$$\max_{x_1, x_2} x_1^{1/3} x_2^{2/3} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = W$$

$$\mathcal{L}(x_1, x_2, \lambda) \equiv x_1^{1/3} \cdot x_2^{2/3} + \lambda \left[W - p_1 x_1 - p_2 x_2 \right]$$

FOC: ① $\frac{\partial \mathcal{L}}{\partial x_1} = 0 \iff \frac{1}{3} x_1^{-2/3} x_2^{2/3} - \lambda p_1 = 0$

② $\frac{\partial \mathcal{L}}{\partial x_2} = 0 \iff \frac{2}{3} x_1^{1/3} x_2^{-1/3} - \lambda p_2 = 0$

③ $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff W - p_1 x_1 - p_2 x_2 = 0$

①/② $\Rightarrow \frac{\frac{1}{3} x_1^{-2/3} x_2^{2/3}}{\frac{2}{3} x_1^{1/3} x_2^{-1/3}} = \frac{1}{2} \frac{x_2}{x_1} = \frac{p_1}{p_2}$

$$\Rightarrow \quad x_2 = 2\frac{p_1}{p_2}x_1$$

SUBSTITUTE INTO (3)

$$\Rightarrow \quad p_1 x_1 + p_2\left(2\frac{p_1}{p_2}\right)x_1 = w$$

$$3p_1x_1 = w$$

$$x_1(p,w) = \frac{w}{3p_1}$$

$$x_2(p,w) = 2\left(\frac{p_1}{p_2}\right)\left(\frac{w}{3p_1}\right) = \frac{2}{3}\frac{w}{p_2}$$

SOC: NOT NEEDED B/C QUASI-CONCAVITY.

DEFN. INDIRECT UTILITY FUNCTION, DENOTED $v(p,w)$, IS THE VALUE FUNCTION FOR THE UMP.

I.E. GIVEN SOLUTION $x(p,w)$ TO UMP

$$\max_{x \in \mathbb{R}^n_+} u(x) \quad \text{s.t.} \quad p \cdot x \leq w,$$

$$v(p,w) \equiv u(x(p,w)).$$

- REMARK: ① DEMAND CORRESPONDENCE $x(p,w)$ GIVES OPTIMAL CONSUMPTION BUNDLE. INDIRECT UTILITY $v(p,w)$ GIVES THE UTILITY LEVEL GAINED FROM CONSUMING $x(p,w)$.

  ② $v(p,w)$ IS THE HIGHEST UTILITY LEVEL THAT CAN BE ACHIEVED WHEN THE CONSUMER HAS WEALTH w AND FACES PRICES p.

PICTURE:

- THM: SUPPOSE $u(\cdot)$ IS A CONT. UTILITY FUNCTION REPRESENTING LOCALLY NON-SATIATED $\succsim$. THE INDIRECT UTILITY FUNCTION $v(p,w)$ IS

① HOMOGENEOUS OF DEGREE ZERO

② STRICTLY INCREASING w AND NON-INCREASING IN p_ℓ $\forall \ell$

③ QUASICONVEX IN (p,w)

④ CONTINUOUS IN p AND w.

EXAMPLE: $u(x_1, x_2) = x_1^{1/3} x_2^{2/3}$

$$x(p,w) = \left(\frac{w}{3p_1}, \frac{2}{3}\frac{w}{p_2} \right)$$

$$v(p,w) = u(x(p,w))$$
$$= \left(\frac{w}{3p_1}\right)^{1/3} \left(\frac{2w}{3p_2}\right)^{2/3} = \left(\frac{4w^3}{27p_1p_2^2}\right)^{1/3}$$
$$= \left(\frac{4}{27p_1p_2^2}\right)^{1/3} w$$

EXERCISE VERIFY THE ABOVE THM.

E-T
FOC'S
I.D

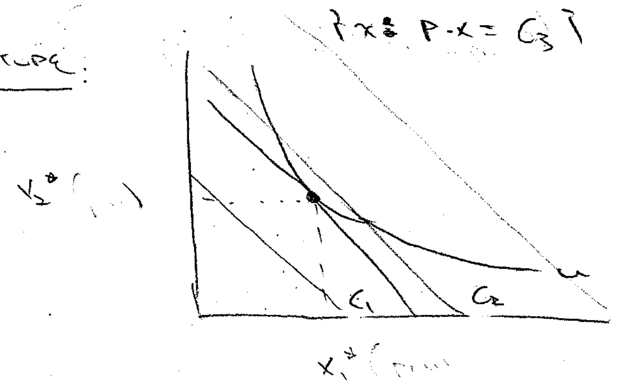
LECT 3

EXPENDITURE MINIMIZATION PROBLEM:

$* \quad \min_{x \in \mathbb{R}^L_+} p \cdot x \quad$ S.T. $u(x) \geq u$

LEAST EXPENSIVE WAY OF ACHIEVING UTILITY LEVEL u GIVEN FACING PRICES p.

PICTURE:



WE'LL ASSUME $p \gg 0$ & $u > 0$

< DO THM RE PROPERTY FIRST >

· EXAMPLE: $u(x_1, x_2) = x_1^{1/3} x_2^{2/3}$

$$\min_{x_1, x_2 \in \mathbb{R}^L_+} p_1 x_1 + p_2 x_2 \quad \text{S.T.} \quad u(x_1, x_2) \geq u.$$

REDUCES TO $\min_{x_1, x_2 \in \mathbb{R}^L} p_1 x_1 + p_2 x_2$ S.T. $u(x_1, x_2) = u$.

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \mu\left[u - x_1^{1/3} x_2^{2/3}\right]$$

FOC ① $p_1 - \mu \frac{1}{3} x_1^{-2/3} x_2^{2/3} = 0$

② $p_2 - \mu \frac{2}{3} x_1^{1/3} x_2^{-1/3} = 0$

③ $u - x_1^{1/3} x_2^{2/3} = 0$ < UTILITY CONSTRAINT >

①/② $= \dfrac{\mu \frac{1}{3} x_1^{-2/3} x_2^{2/3}}{\mu \frac{2}{3} x_1^{1/3} x_2^{-1/3}} = \dfrac{1}{2}\dfrac{x_2}{x_1} = \dfrac{p_1}{p_2}$ < MRS = PRICE RATIO >

$\Rightarrow \quad x_2 = 2\frac{p_1}{p_2} x_1$

SUBST INTO U-CONSTRAINT ③

$\Rightarrow \quad x_1^{1/3}\left(2\frac{p_1 x_1}{p_2}\right)^{2/3} = u$

$x_1 \left(2\frac{p_1}{p_2}\right)^{2/3} = u$

$h(p,u) \equiv x_1^* = \left(\frac{p_2}{2p_1}\right)^{2/3} u$

$x_2^* = 2\frac{p_1}{p_2}\left(\frac{p_2}{2p_1}\right)^{2/3} u$

$h(p,u) = x_2^* = \left(2\frac{p_1}{p_2}\right)^{1/3} u$

SOLUTION TO EMP IS CALLED HICKSIAN DEMAND

THM: SUPPOSE $u(\cdot)$ IS A CONT. UTILITY FUNCTION REPRESENTING LOCALLY NON-SATIATED $\succsim$. THEN FOR ANY $p \gg 0$, HICKSIAN DEMAND SATISFIES

① HD0 IN p: $h(\alpha p, u) = h(p,u)$ $\forall p, u$ AND $\alpha > 0$

② NO EXCESS UTILITY: $\forall x \in h(p,u)$, $u(x) = u$

③ CONVEXITY / UNIQUENESS: IF $\succsim$ IS CONVEX THEN $h(p,u)$ IS A CONVEX SET. IF $\succsim$ IS S. CONVEX THEN $h(p,u)$ IS A SINGLETON.

- DEFN: THE EXPENDITURE FUNCTION IS THE VALUE FUNCTION FOR THE EMP.

I.E. GIVEN EMP $\min_{x \in \mathbb{R}^L_+} p \cdot x$ S.T. $u(x) \geq u$

AND ITS SOLUTION $h(p,u)$, $e(p,u) = p \cdot h(p,u)$

- EXAMPLE: $u(x) = x_1^{1/3} x_2^{2/3}$, HAD

$$h(p,u) = \left(\left(\frac{p_2}{2p_1}\right)^{2/3} u,\ \left(\frac{2p_1}{p_2}\right)^{1/3} u\right)$$

So
$$\begin{aligned}
e(p,u) &= p_1 \left(\frac{p_2}{2p_1}\right)^{2/3} u + p_2\left(\frac{2p_1}{p_2}\right)^{1/3} u \\
&= \left[\left(\tfrac{1}{2}\right)^{2/3} p_2^{2/3} p_1^{1/3} + 2^{1/3} p_1^{1/3} p_2^{2/3}\right] u \\
&= \left[\left(\tfrac{1}{2}\right)^{2/3} + 2^{1/3}\right] p_1^{1/3} p_2^{2/3} u \\
&= \left[\tfrac{1}{2}^{2/3} + 2\left(\tfrac{1}{2}^{2/3}\right)\right] p_1^{1/3} p_2^{2/3} u \\
&= 3\left(\tfrac{1}{2}\right)^{2/3} p_1^{1/3} p_2^{2/3} u.
\end{aligned}$$

- THM: SUPPOSE $u(\cdot)$ IS A CONT UTILITY FUNC REPRES. LOCALLY NON-SATIATED $\succsim$. THEN EXPENDITURE FUNC $e(p,u)$ IS

① HD1 IN p

② STRICTLY INCREASING IN u AND NON-DECREASING IN p_ℓ FOR ALL ℓ.

③ CONCAVE IN p

④ CONTINUOUS IN p & u

• OF THE ~~FOUR~~ PROPERTIES, OF $e(p,u)$, CONCAVITY IN p IS OF PARTICULAR INTEREST. (TO BE SEEN LATER)

TO SEE THIS: TAKE ANY p, p' & $\alpha > 0$.

WE NEED TO SHOW

$$e(\alpha p + (1-\alpha)p', u) \geq \alpha e(p,u) + (1-\alpha) e(p',u).$$

LET $p'' = \alpha p + (1-\alpha) p'$

$e(p,u) \leq p \cdot$ ANY x THAT ACHIEVES UTILITY u

$h(p'',u)$ ACHIEVES UTILITY LEVEL u.

SO, $e(p,u) \leq p \cdot h(p'',u)$

SIMILARLY $e(p',u) \leq p' \cdot h(p'',u)$

$$\begin{aligned} \text{So, } \alpha e(p,u) + (1-\alpha) e(p',u) &\leq \alpha p \cdot h(p'',u) + (1-\alpha) p' \cdot h(p'',u) \\ &= (\alpha p + (1-\alpha)p') \cdot h(p'',u) \\ &= p'' \cdot h(p'',u) \\ &= e(p'',u). \end{aligned}$$

-//-

COROLLARY. SUPPOSE U(·) CONT. REP. LNS $\succsim$. $h(p,u)$ IS A SINGLETON $\forall p \gg 0$. THEN $h(p,u)$ SATISFIES THE COMPENSATED LAW OF DEMAND:

I.E. $\forall p', p''$, HAVE $(p''-p') \cdot (h(p'',u) - h(p',u)) \leq 0$

E.G. $p_i'' > p_i'$ & $p_j'' = p_j'$ $\forall j \neq i$

THEN $(p''-p') \cdot (h(p'',u) - h(p',u)) = (p_i''-p_i')(h(p'',u$

- **RELATIONSHIP** BETWEEN $x(p,w)$ & $h(p,u)$

THM. SUPPOSE $u(\cdot)$ IS CONT. UTILITY FUNC. REPRES. LNS $\succsim$ & $p \gg 0$. THEN WE HAVE

SUPPOSE $w > 0$.

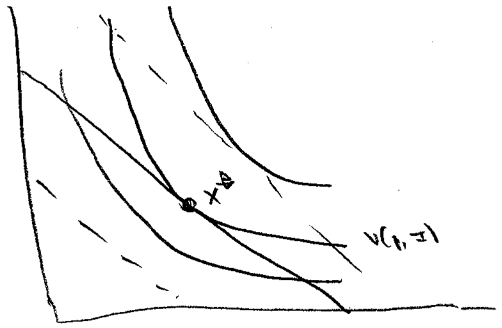
① $x(p,w) = h(p, v(p,w))$ $\qquad e(p, v(p,w)) = w$

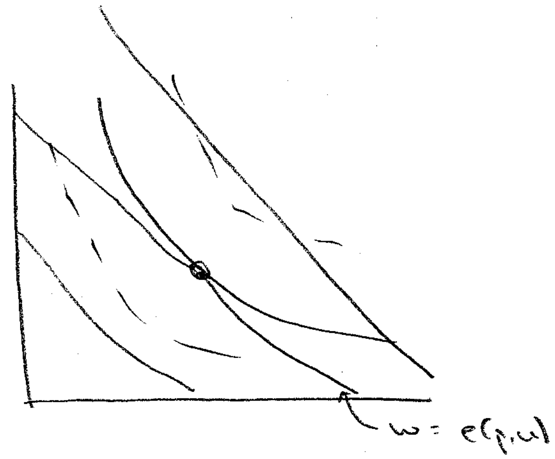
SUPPOSE $u > u(0)$

② $h(p,u) = x(p, e(p,u))$ $\qquad v(p, e(p,u)) = u$

SKETCH:

①





- ~~ENVELOPE THM.~~

~~GIVEN $\underset{x}{\text{OPT}}\ f(x;\alpha)$ S.T. $g(x;\alpha) = 0$~~

~~& ASSOCIATED LAGRANGIAN~~

~~$\mathcal{L} = f(x) + \lambda g(x)$, WE HAVE~~

~~$\dfrac{\partial V(\alpha)}{\partial \alpha_i} = \dfrac{\partial \mathcal{L}}{\partial \alpha_i}\Big|_{x^*(\alpha),\ \lambda^*(\alpha)}$~~

ENVELOPE THM.

$\max_x f(x;\alpha)$, $f: \mathbb{R}^2 \to \mathbb{R}$

$x \in \mathbb{R}$

$\alpha \in \mathbb{R}$

SOLN $x(\alpha) \equiv x^*$

ENV. THM

VALUE FUNCTION $V(\alpha) = f(x(\alpha);\alpha)$ $\frac{\partial V(\alpha)}{\partial \alpha_i} = \frac{\partial f}{\partial \alpha_i}\Big|_{x(\alpha)}$

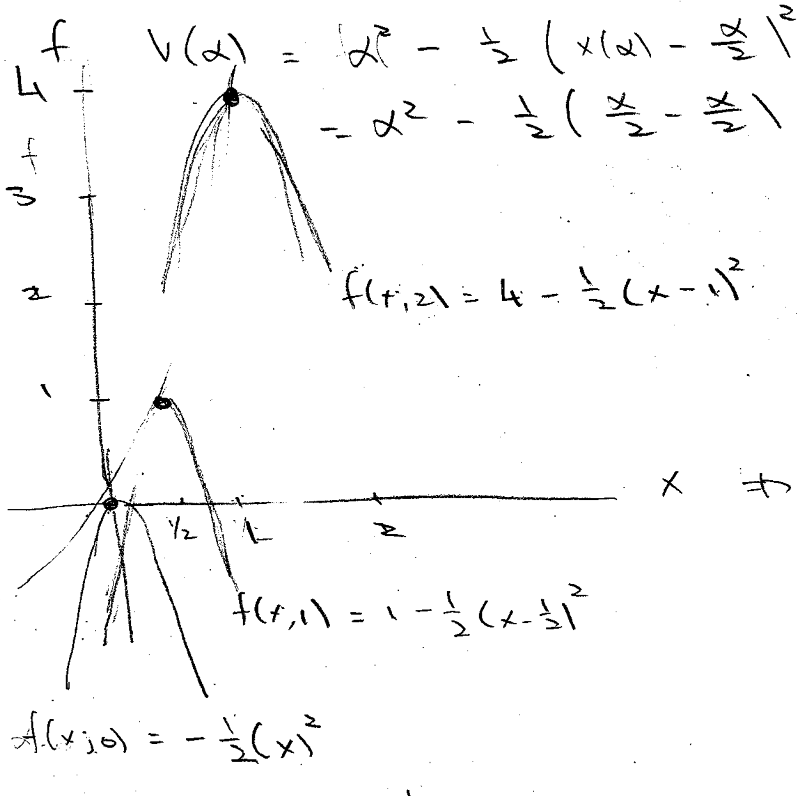
EXAMPLE: $f(x;\alpha) = \alpha^2 - \frac{1}{2}\left(x - \frac{\alpha}{2}\right)^2$

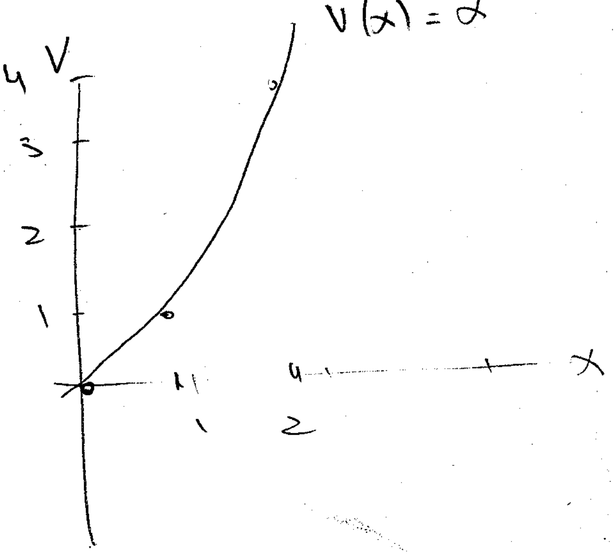
$\max_x \alpha^2 - \frac{1}{2}\left(x - \frac{\alpha}{2}\right)^2$

FOC: $-\left(x - \frac{\alpha}{2}\right) = 0 \Rightarrow x(\alpha) = \frac{\alpha}{2}$

$V(\alpha) = \alpha^2 - \frac{1}{2}\left(x(\alpha) - \frac{\alpha}{2}\right)^2$

$= \alpha^2 - \frac{1}{2}\left(\frac{\alpha}{2} - \frac{\alpha}{2}\right) = \alpha^2$





NOTE: $\frac{dV(\alpha)}{d\alpha} = 2\alpha =$

AND $\frac{\partial f}{\partial \alpha}\Big|_{x(\alpha)} = 2\alpha - \left(x(\alpha) - \frac{\alpha}{2}\right)\left(-\frac{1}{2}\right)$

$= 2\alpha + \frac{1}{2}\left(\frac{\alpha}{2} - \frac{\alpha}{2}\right) = 2\alpha.$

I.E. VERIFIED $\frac{dV(\alpha)}{d\alpha} = \frac{\partial f}{\partial \alpha}\Big|_{x(\alpha)}$ AS CLAIMED.

THIS MAY NOT SEEM VERY USEFUL.
SO, SLIGHTLY STRONGER VERSION.

ENV. THM: GIVEN OPT. PROBLEM

$$\underset{x}{\text{OPT}}\ f(x;\alpha) \quad \text{S.T.} \quad g(x;\alpha) = 0,$$

WHERE $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^m$ & $f: \mathbb{R}^{n+m} \to \mathbb{R}$, $g: \mathbb{R}^{n+m} \to \mathbb{R}$

LET $\mathcal{L}(x, \lambda; \alpha) = f(x;\alpha) + \lambda g(x;\alpha)$

BE THE ASSOCIATED LAGRANGIAN.

THEN $\dfrac{\partial V(\alpha)}{\partial \alpha_i} = \dfrac{\partial \mathcal{L}}{\partial \alpha_i}\Big|_{x(\alpha), \lambda(\alpha)}$.

-11-

ROY'S IDENTITY: SUPPOSE $u(\cdot)$ IS CTF REPRES. L.N.S & STRICTLY CONVEX $\succsim$. SUPPOSE $V(p,w)$ IS DIFF. AT $(p,w) \gg 0$.

THEN $$x_\ell(p,w) = - \frac{\frac{\partial V(p,w)}{\partial p_\ell}}{\frac{\partial V(p,w)}{\partial w}} \quad \forall \ell.$$

I.E $$x(p,w) = - \frac{1}{\underbrace{\nabla_w V(p,w)}_{\text{SCALAR}}} \nabla_p V(p,w)$$

TO SEE: $\max_x u(x)$ S.T. $p \cdot x = w$

$$\mathcal{L} = u(x) + \lambda [w - p \cdot x]$$

$$\frac{\partial V}{\partial p_\ell} = \frac{\partial \mathcal{L}}{\partial p_\ell}\bigg|_{x^*, \lambda^*} = -\lambda^* x_\ell^*$$

$$\frac{\partial V}{\partial w} = \frac{\partial \mathcal{L}}{\partial w}\bigg|_{x^*, \lambda^*} = +\lambda^*$$

SO, $$-\frac{\frac{\partial V}{\partial p_\ell}}{\frac{\partial V}{\partial w}} = -\frac{-\lambda^* x_\ell^*}{\lambda^*} = x_\ell^* \equiv x_\ell(p,w).$$

—||—

[SHEPARD'S LEMMA]

THM: SUPPOSE $u(\cdot)$ IS A CONT. UTILITY FUNC REPRES. L.N-S & STRICTLY CONVEX $\succsim$. THEN

$$h(p,u) = \nabla_p e(p,u)$$

I.E. $h_i(p,u) = \dfrac{\partial e(p,u)}{\partial p_i} \quad \forall i.$

PF: RECALL $e(p,u)$ IS THE VALUE FUNCTION FOR $\min_x \; p \cdot x$ S.T. $u(x) = u$.

ASSOCIATED $\mathcal{L}$ IS

$$\mathcal{L} = p \cdot x + \mu\,[u - u(x)]$$

BY THE ENV. THM,

$$\frac{\partial e(p,u)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}\Big|_{x^*, \lambda^*}$$

$$= x_i\Big|_{x^*, \lambda^*} + 0$$

$$= x_i^* = h_i(p,u)$$

—//—

THM: SUPPOSE $u(\cdot)$ IS A CONT. UTILITY FUNC REPRES. A L.N-S & STRICTLY CONVEX $\succsim$. SUPPOSE $h(p,u)$ IS CONT. DIFFERENTIABLE AT (p,u). THEN LETTING

$$D_p h(p,u) = \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \frac{\partial h_1}{\partial p_2} & \cdots & \frac{\partial h_1}{\partial p_n} \\ \vdots & & & \\ \frac{\partial h_n}{\partial p_1} & \frac{\partial h_n}{\partial p_2} & \cdots & \frac{\partial h_n}{\partial p_n} \end{bmatrix},$$

WE HAVE

1. $D_p h(p,u) = D_p^2 e(p,u)$
2. $D_p h(p,u)$ IS N.S.D $\langle y^T [D_p h(p,u)] y \le 0 \;\; \forall y \rangle$
3. $D_p h(p,u)$ IS SYMMETRIC
4. $D_p h(p,u)\, p = 0$ $\langle$ TO SEE: DIFF. $h(\alpha p,u) - h(p,u)$ W.R.T. $\alpha \rangle$

REMARK:

1) NSD OF $D_p h(p,u) \Rightarrow$ LAW OF COMP. DEMAND

LET $y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ℓ-th place.

THEN
$$y^T D_p h(p,u) y = [0 \cdots 0 1 0 \cdots 0] \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_i} & \cdots & \frac{\partial h_1}{\partial p_n} \\ \vdots & & & & \\ \frac{\partial h_n}{\partial p_1} & \cdots & \frac{\partial h_n}{\partial p_i} & \cdots & \frac{\partial h_n}{\partial p_n} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial h_i}{\partial p_1} & \frac{\partial h_i}{\partial p_2} & \cdots & \frac{\partial h_i}{\partial p_n} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{\partial h_i}{\partial p_i} \underset{\text{B/C NSD}}{\leq} 0$$

2) SYMMETRY?

- HICKSIAN DEMAND IS BETTER BEHAVED THEN MARSHALLIAN DEMAND B/C LAW OF DEMAND.
BUT HICKSIAN DEMAND IS UNOBSERVABLE. NEVERTHELESS, FOLLOWING THM SHOW THAT $\frac{\partial h_\ell}{\partial p_k}$ CAN BE FOUND:

THM [SLUTSKY EQUATION]: SUPPOSE $u(\cdot)$ IS A CONT. UTILITY FUNCTION REPRESENT A $\succsim$ THAT IS L.N-S & S. CONVEX. THEN $\forall (p,w)$ & $u = v(p,w)$, WE HAVE

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w) \quad \forall \ell, k$$

I.E. $D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T$

To SEE: $\frac{\partial}{\partial p_k}\left[h_\ell(p,\bar{u})\right] = \frac{\partial}{\partial p_k}\left[x_\ell(p, e(p,u))\right]$

$$\Rightarrow \frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial}{\partial p_k} x_\ell(p, e(p,u))$$

$$= \frac{\partial x_\ell(p,e(p,u))}{\partial p_k} + \frac{\partial x_\ell(p,e(p,u))}{\partial w}\frac{\partial e(p,u)}{\partial p_k}$$

$$= \frac{\partial x_\ell(p,e(p,u))}{\partial p_k} + \frac{\partial x_\ell(p,e(p,u))}{\partial w} h_k(p,u)$$

SINCE $u = v(p,w)$, $e(p,u) = w$ & $h(p,u) = x(p, e(p,u)) = x(p,w)$

$$\Rightarrow \frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)$$

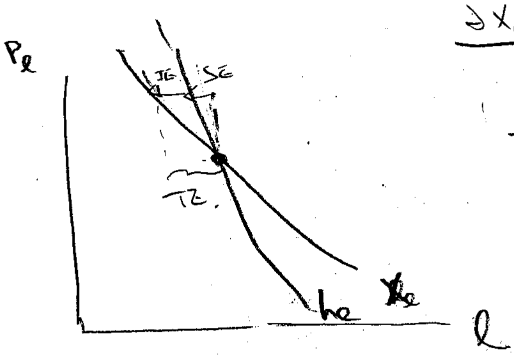
—//—

REMARK: $\underbrace{\frac{\partial x_\ell(p,w)}{\partial p_k}}_{TE} = \underbrace{\frac{\partial h_\ell(p,u)}{\partial p_k}}_{SE} \underbrace{- \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)}_{IE}$

$TE = SE + IE$.

IF $\frac{\partial x_\ell}{\partial w} > 0$, I.E. GOOD ℓ IS NORMAL.

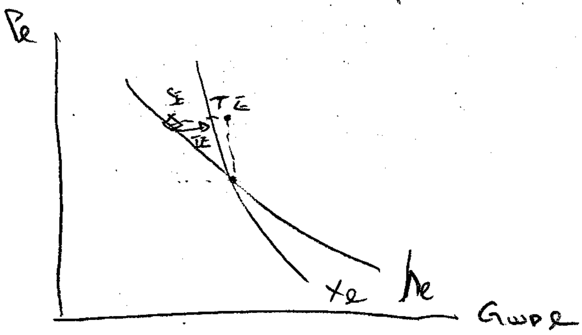
$$\frac{\partial x_\ell(p,w)}{\partial p_k} < \frac{\partial h_\ell(p,u)}{\partial p_k}$$



$\Rightarrow x_\ell$ IS FLATTER THAN h_ℓ ON p_ℓ / x_ℓ GRAPH.

IF $\frac{\partial x_\ell}{\partial w} < 0$, I.E. GOOD ℓ IS INFERIOR

$$\frac{\partial x_\ell(p,w)}{\partial p_k} > \frac{\partial h_\ell(p,u)}{\partial p_k}$$



$\Rightarrow x_\ell$ IS STEEPER THAN h_ℓ ON p_ℓ / x_ℓ GRAPH

REMARK: $D_p h(p,u) \equiv$ SLUTSKY MATRIX $= S(p,w)$.

DIRECTLY COMPUTABLE FROM $X(p,w)$.

GENERAL REMARK:

WE HAVE SEEN THAT IF A DEMAND FUNCTION $X(p,w)$ IS GENERATED BY A RATIONAL U-MAXIMIZER, THEN IT MUST SATISFY

1. HD ϕ
2. W. LAW
3. HAVE A $S(p,w)$ THAT IS SYMM & NSD

THERE ARE RESULTS <INTEGRABILITY RESULTS> SHOW THAT IF DEMAND FUNCTION $X(p,w)$ SATISFY THESE 1-3, THEN THERE IS A RATIONAL UTILITY FUNCTION u THAT GENERATES THIS DEMAND FUNCTION

I.E. 1-3 ARE THE ONLY IMPLICATIONS OF THE DEMAND THEORY