Advanced Microeconomics I

Fall 2023 - M. Pak

Problem Set 5: Suggested Solutions

1. Consider an Edgeworth box economy where preferences are given by

$$u_1(x_{11}, x_{21}) = x_{11} + \ln x_{21} \quad \text{and} \quad u_2(x_{12}, x_{22}) = x_{12}x_{22},$$

and the initial endowments are

$$\omega_1 = (1,3) \quad \text{and} \quad \omega_2 = (3,1).$$

(a) Using the normalization $p_2 = 1$, Find all the Walrasian equilibrium. You may assume that the solution is interior.

Solution: To find consumer 1's Marshallian demand, we solve

$$\max_{x_{11}, x_{21}} \; x_{11} + \ln x_{21} \quad \text{s.t.} \quad p_1 x_{11} + p_2 x_{21} = p_1 + 3p_2$$

Setting MRS = price ratio yields

$$x_{21} = \frac{p_1}{p_2}.$$

Substituting this into (3) yields:

$$p_1 x_{11} + p_2 \left(\frac{p_1}{p_2} \right) = p_1 + 3p_2 \quad \Rightarrow \quad x_{11} = \frac{3p_2}{p_1}.$$

Note that this also shows that there are no boundary solutions in this case. So, Marshallian demand for consumer 1 is:

$$x_1(p_1, p_2) = \left(\frac{3p_2}{p_1}, \frac{p_1}{p_2} \right).$$

To find consumer 2's Marshallian demand function, we note that her utility function is a Cobb-Douglas utility function, so the demand function is given by

$$x_2(p_1, p_2) = \left(\frac{3p_1 + p_2}{2p_1}, \frac{3p_1 + p_2}{2p_2} \right).$$

Now, we find the equilibrium prices by normalizing $p_2 = 1$ and looking for market clearing prices for the market for good 2:

$$x_{21}(p_1^*, 1) + x_{22}(p_1^*, 1) = \omega_{21} + \omega_{22} \Longrightarrow \frac{p_1^*}{1} + \frac{3p_1^* + 1}{2} = 4$$

$$\Longrightarrow p_1^* = \frac{7}{5}.$$

So, the Walrasian equilibrium price vector is $p^* = (\frac{7}{5}, 1)$, and the corresponding Walrasian equilibrium allocation is:

$$x_1^* = \left(\frac{15}{7}, \frac{7}{5}\right) \quad \text{and} \quad x_2^* = \left(\frac{13}{7}, \frac{13}{5}\right).$$

(b) Verify that the first welfare theorem holds.

Solution: At the Walrasian equilibrium allocation, we have

$$\text{MRS}_1\big|_{x_1^*} = \frac{x_{21}^*}{1} = \frac{7}{5} \quad \text{and} \quad \text{MRS}_2\big|_{x_1^*} = \frac{x_{22}^*}{x_{12}^*} = \frac{\frac{13}{5}}{\frac{13}{7}} = \frac{7}{5}.$$

Since $\text{MRS}_1\big|_{x_1^*} = \text{MRS}_2\big|_{x_2^*}$, (x_1^*, x_2^*) is Pareto optimal, as required.

(c) Can the allocation $\hat{x} = (\hat{x}_1, \hat{x}_2) = ((1,1),(3,3))$ be supported as a Walrasian equilibrium with transfers? If yes, find the supporting prices and transfers. If no, explain.

Solution: At allocation $\hat{x}$, we have

$$\begin{aligned} MRS_1 &= \hat{x}_{21} = 1 \\ MRS_2 &= \frac{\hat{x}_{22}}{\hat{x}_{12}} = \frac{3}{3} = 1 \\ \Longrightarrow MRS_1 &= MRS_2. \end{aligned}$$

Thus, $\hat{x}$ is Pareto optimal, and the Second Welfare Theorem implies that we can support this allocation as an equilibrium with transfers. The supporting prices are given by the MRS of the consumers at this allocation. That is, $\frac{\hat{p}_1}{\hat{p}_2} = 1$. With our normalization, we get $\hat{p} = (1,1)$. So, for transfers,

$$\begin{aligned} T_1 &= (\hat{p}_1, \hat{p}_2)\cdot(\hat{x}_{11}, \hat{x}_{21}) - (\hat{p}_1, \hat{p}_2)\cdot(\omega_{11}, \omega_{21}) \\ &= (1,1)\cdot(1,1) - (1,1)\cdot(1,3) = -2 \\ T_2 &= -T_1 = 2. \end{aligned}$$

2. Consider a Robinson Crusoe economy where the consumer's utility function is $u(x_1, x_2) = 2\ln x_1 + \ln x_2$ and the production function is $f(z) = z^{\frac{1}{2}}$. Suppose the consumer's endowment is $\omega = (5, 0)$.

(a) Letting w and p_2 be the prices of good 1 and good 2, respectively, find the firm's unconditional input demand, supply, and profit functions.

Solution: The firm's profit maximization problem is:

$$\max_z \; p_2 z^{\frac{1}{2}} - wz$$

$$\text{FOC is:} \quad \frac{p_2}{2} z^{-\frac{1}{2}} - w = 0 \quad \Longrightarrow \quad z(p_2, w) = \frac{p_2^2}{4w^2}$$

$$y(p_2, w) = \left(\frac{p_2^2}{4w^2}\right)^{\frac{1}{2}} = \frac{p_2}{2w}$$

$$\pi(p_2, w) = p_2 \frac{p_2}{2w} - w\frac{p_2^2}{4w^2} = \frac{p_2^2}{4w}.$$

(b) As usual, assume that all the profits of the firm goes to the consumer in this economy and find find the consumer's demand function.

Solution: In this example, the consumer's demand can be easily found by transforming the utility function into a standard Cobb-Douglas form and applying the usual formula. However, as a practice, we'll find the demand by actually solving the optimization problem. The consumer's problem is:

$$\max_{x_1,x_2} 2\ln x_1 + \ln x_2 \quad \text{s.t.} \quad wx_1 + p_2x_2 \le w\omega_1 + p_2\omega_2 + \pi(p_2,w).$$

$$\implies \quad \mathscr{L} = 2\ln x_1 + \ln x_2 + \lambda\left(5w + \frac{p_2^2}{4w} - wx_1 - p_2x_2\right).$$

Assuming interior solution:

$$(1) \qquad \frac{\partial\mathscr{L}}{\partial x_1} = \frac{2}{x_1} - \lambda w = 0$$

$$(2) \qquad \frac{\partial\mathscr{L}}{\partial x_2} = \frac{1}{x_2} - \lambda p_2 = 0$$

$$(3) \qquad \frac{\partial\mathscr{L}}{\partial \lambda} = 5w + \frac{p_2^2}{4w} - wx_1 - p_2x_2 = 0.$$

From (1) and (2) we can get:

$$\frac{2x_2}{x_1} = \frac{w}{p_2} \implies x_1 = \frac{2p}{w}x_2$$

Substituting this into (3) yields:

$$w\left(\frac{2p}{w}x_2\right) + p_2x_2 = 5w + \frac{p_2^2}{4w}$$

$$\implies \quad x_2(w,p_2) = \frac{5w + \frac{p_2^2}{4w}}{3p} \quad \text{and} \quad x_1(w,p_2) = \frac{2\left(5w + \frac{p_2^2}{4w}\right)}{3w}.$$

(c) Suppose prices are $(w,p_2) = (1,1)$. Using a single diagram, graph the consumer's utility-maximizing consumption bundle and the firm's profit maximizing production plan. Is this a Walrasian equilibrium price vector? Explain.

Solution: When $(w,p_2) = (1,1)$, $z(1,1) = \frac{1}{4}$ and $y(1,1) = \frac{1}{2}$, while $x_1(1,1) = \frac{7}{2}$ and $x_2(1,1) = \frac{7}{4}$. The graph of the economy is given in figure 1. As this figure shows, at prices $(w,p_2) = (1,1)$, markets do not clear. For example,

$$x_2(1,1) = \frac{7}{4} \ne 0 + \frac{1}{2} = \omega_2 + y(1,1).$$

(d) Find all the Walrasian equilibrium using the normalization $p_2 = 1$.

Solution: We solve for the market clearing condition for good 2:

$$x_2(w,1) = \omega_2 + y(w,1) \quad \implies \quad \frac{5w + \frac{1}{4w}}{3} = \frac{1}{2w} \quad \implies \quad w^* = \frac{1}{2}.$$

Thus, Walrasian equilibrium price is $(w^*, p_2^*) = (\frac{1}{2}, 1)$, and the corresponding Walrasian equilibrium consumption and production plans are $(x_1^*, x_2^*) = (4,1)$ and $(z^*, y^*) = (1,1)$.

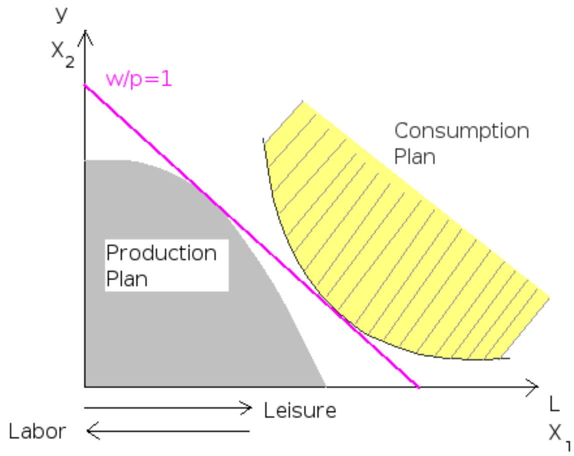

Figure 1: Production and Consumption Plan at $(w, p_2) = (1, 1)$

3. Consider a 2×2 production model where the two firms' production functions are

$$f_1(z_{11}, z_{21}) = z_{11}^{\frac{2}{3}} z_{21}^{\frac{1}{3}} \quad \text{and} \quad f_2(z_{12}, z_{22}) = z_{12}^{\frac{1}{2}} z_{22}^{\frac{1}{2}}.$$

The aggregate factor endowments are $\bar{z}_1 = 10$ and $\bar{z}_2 = 10$, and the prices of the output goods are $p_1 = 10$ and $p_1 = 10$.

(a) Verify that the production of good 1 is relatively more intense than the production of good 2.

Solution: Recall that for a generic CRS Cobb-Douglas production function,

$$f_j(z_{1j}, z_{2j}) = z_{1j}^{a_j} z_{2j}^{b_j},$$

we have

$$z_{1j}(w_1^*, w_2^*, 1) = \left(\frac{a_j w_2}{b_j w_1}\right)^{b_j} q \quad \text{and} \quad z_{2j}(w_1^*, w_2^*, 1) = \left(\frac{b_j w_1}{a_j w_2}\right)^{a_j} q.$$

$$\Longrightarrow \frac{z_{1j}(w_1, w_2, 1)}{z_{2j}(w_1, w_2, 1)} = \frac{\left(\frac{a_j w_2}{b_j w_1}\right)^{b_j} q}{\left(\frac{b_j w_1}{a_j w_2}\right)^{a_j} q} = \left(\frac{a_j w_2}{b_j w_1}\right)^{b_j} \left(\frac{a_j w_2}{b_j w_1}\right)^{a_j} = \frac{a_j w_2}{b_j w_1},$$

since $a_j + b_j = 1$ in our case. Therefore,

$$\frac{z_{11}(w_1, w_2, 1)}{z_{21}(w_1, w_2, 1)} = \frac{\frac{2}{3} w_2}{\frac{1}{3} w_1} = \frac{2 w_2}{w_1} > \frac{w_2}{w_1} = \frac{\frac{1}{2} w_2}{\frac{1}{2} w_1} = \frac{z_{12}(w_1, w_2, 1)}{z_{22}(w_1, w_2, 1)}.$$

(b) Find the equilibrium factor prices.

Solution: To find the candidates for factor market equilibrium prices, we solve the two zero-profit conditions. The unit cost function of a firm with standard Cobb-Douglas production function is

$$c_j(w_1, w_2, 1) = \left[\left(\frac{a_j}{b_j}\right)^{b_j} + \left(\frac{b_j}{a_j}\right)^{a_j}\right] w_1^{a_j} w_2^{b_j}.$$

Thus, the zero profit condition for firm 1 yields

$$10 = \left[\left(\frac{\frac{2}{3}}{\frac{1}{3}} \right)^{\frac{1}{3}} + \left(\frac{\frac{1}{3}}{\frac{2}{3}} \right)^{\frac{2}{3}} \right] w_1^{\frac{2}{3}} w_2^{\frac{1}{3}} = \left[2^{\frac{1}{3}} + \left(\frac{1}{2} \right)^{\frac{2}{3}} \right] w_1^{\frac{2}{3}} w_2^{\frac{1}{3}} = 1.890\, w_1^{\frac{2}{3}} w_2^{\frac{1}{3}}$$

$$\implies w_1^2 w_2 = \left(\frac{10}{1.890} \right)^3 = 148.12 \implies w_2 = \frac{148.12}{w_1^2}.$$

The zero profit condition for firm 2 yields

$$10 = \left[\left(\frac{\frac{1}{2}}{\frac{1}{2}} \right)^{\frac{1}{2}} + \left(\frac{\frac{1}{2}}{\frac{1}{2}} \right)^{\frac{1}{2}} \right] w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} = 2 w_1^{\frac{1}{2}} w_2^{\frac{1}{2}}$$

$$\implies w_1 w_2 = \left(\frac{10}{2} \right)^2 = 25 \implies w_1 \left(\frac{148.12}{w_1^2} \right) = 25$$

$$w_1^* = \left(\frac{148.12}{25} \right) = 5.92$$

$$w_2^* = \frac{148.12}{5.92^2} = 4.23.$$

To be sure that this the equilibrium, we need to check that the ratio of the aggregate factor endowments. $\frac{\bar{z}_1}{\bar{z}_2} = \frac{10}{10} = 1$ fall between the two firms' factor intensity ratios at these prices:

$$\frac{z_{11}(w_1^*, w_2^*, 1)}{z_{21}(w_1^*, w_2^*, 1)} = \frac{2w_2^*}{w_1^*} = \frac{2(4.23)}{5.92} = 1.43 > 1 > 0.71 = \frac{4.23}{5.92} = \frac{w_2^*}{w_1^*} = \frac{z_{12}(w_1^*, w_2^*, 1)}{z_{22}(w_1^*, w_2^*, 1)},$$

as required.

(c) Find the equilibrium factor allocation. What are the output levels in the equilibrium?

Solution: Using the conditional factor demand from part (a), we have,

$$z_{11}(w_1^*, w_2^*, 1) = \left(\frac{\frac{2}{3}(4.23)}{\frac{1}{3}(5.92)} \right)^{\frac{1}{3}} = \left(\frac{2(4.23)}{5.92} \right)^{\frac{1}{3}} = 1.126$$

$$z_{21}(w_1^*, w_2^*, 1) = \left(\frac{\frac{1}{3}(5.92)}{\frac{2}{3}(4.23)} \right)^{\frac{2}{3}} = \left(\frac{5.92}{2(4.23)} \right)^{\frac{2}{3}} = 0.788$$

$$z_{12}(w_1^*, w_2^*, 1) = \left(\frac{\frac{1}{2}(4.23)}{\frac{1}{2}(5.92)} \right)^{\frac{1}{2}} = \left(\frac{4.23}{5.92} \right)^{\frac{1}{2}} = 0.845$$

$$z_{22}(w_1^*, w_2, 1) = \left(\frac{\frac{1}{2}(5.92)}{\frac{1}{2}(4.23)} \right)^{\frac{1}{2}} = \left(\frac{5.92}{4.23} \right)^{\frac{1}{2}} = 1.183.$$

Next, we find the scaling required to clear the factor markets. That is we find, $\alpha > 0$ and $\beta > 0$ so that $\alpha z(w_1^*, w_2^*, 1) + \beta z(w_1^*, w_2^*, 1) = \bar{z}$. We write this as a column vector equation to make the scaling clearer.

$$\alpha \begin{bmatrix} 1.126 \\ 0.788 \end{bmatrix} + \beta \begin{bmatrix} 0.845 \\ 1.183 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

The first equation yields

$$\alpha = \frac{10 - 0.845\beta}{1.126} = 8.881 - 0.750\beta.$$

Substitute this into the second equation to obtain

$$\begin{aligned}
&(8.881 - 0.750\beta)(0.788) + 1.183\beta = 10 \\
&\qquad \Longrightarrow \beta = \frac{10 - 8.881(0.788)}{1.183 - 0.750(0.788)} = \frac{3.002}{0.592} = 5.071 \\
&\qquad \alpha = 8.881 - 0.750(5.071) = 5.078.
\end{aligned}$$

Therefore, the equilibrum output levels are $q_1^* = \alpha = 5.071$ and $q_2^* = \beta = 5.078$. And the equilibrium factor allocations are:

$$\begin{aligned}
z_{11}(w_1^*, w_2^*, q_1^*) &= 5.071(1.126) = 5.710 \\
z_{21}(w_1^*, w_2^*, q_1^*) &= 5.071(0.788) = 3.996 \\
z_{12}(w_1^*, w_2^*, q_1^*) &= 5.078(0.845) = 4.291 \\
z_{22}(w_1^*, w_2^*, q_1^*) &= 5.078(1.183) = 6.007.
\end{aligned}$$

To verify that the markets indeed clear, note that $z_{11}^* + z_{12}^* = 5.710 + 4.291 = 10.001 \approx \bar{z}_1$ and $z_{21}^* + z_{22}^* = 3.996 + 6.007 = 10.003 \approx 10 = \bar{z}_2$. The discrepancies are rounding errors.