Advanced Microeconomics I

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Problem Set 4: Suggested Solutions

1. Show that $c(F,u_2) \leq c(F,u_1)$ for all distributions F if and only if there exists an increasing concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1(x))$. (Hint: use Jensen's inequality).

   Solution: Any concave function $g(\cdot)$ satisfies Jensen's inequality:

$$\int g(x)\,dF(x) \leq g\left(\int x\,dF(x)\right).$$

   For any distribution F, we have

$$\begin{aligned} u_2(c(F,u_2)) &= \int u_2(x)\,dF && \text{by the definition of certainty equivalent} \\ &= \int \phi(u_1(x))\,dF \\ &\leq \phi\left(\int u_1(x)\,dF\right) && \text{by Jensen's inequality} \\ &= \phi(u_1(c(F,u_1))) && \text{by the definition of certainty equivalent} \\ &= u_2(c(F,u_1)). \end{aligned}$$

   Since $u_2(c(F,u_2)) \leq u_2(c(F,u_1))$ and $u_2(\cdot)$ is increasing, we have $c(F,u_2) \leq c(F,u_1)$ as desired.

2. A strictly risk-averse decision maker has initial wealth W and faces possible loss of $D < W$. The probability that the loss will occur is $\frac{1}{2}$. Suppose insurance is available at price $q < 1$ per unit, where q is not necessarily the fair price.

   (a) For what values of q will the decision maker want to buy strictly positive amount of insurance?

   Solution: The DM's expected utility is

$$\begin{aligned} U(x) &= \frac{1}{2}u(W-qx) + \frac{1}{2}u(W-D-qx+x) \\ &= \frac{1}{2}u(W-qx) + \frac{1}{2}u(W-D+(1-q)x). \\ \Rightarrow \quad MU(x) &= \frac{1}{2}u'(W-qx)(-q) + \frac{1}{2}u'(W-D+(1-q)x)(1-q). \end{aligned}$$

For $x^* > 0$, we need require $MU(0) > 0$:

$$\begin{aligned}\frac{1}{2}u'(W)(-q)+\frac{1}{2}u'(W-D)(1-q) &> 0\\ \Rightarrow \quad q &< \frac{u'(W-D)}{u'(W)+u'(W-D)}.\end{aligned}$$

(b) Suppose the decision maker exhibits decreasing absolute risk aversion. Assuming that the optimal insurance purchase x^* is an interior solution (that is, $x^* \in (0,D)$) determine whether x^* is increasing or decreasing as the initial wealth, W, increases.

Solution: FOC for interior solution is given by

$$\frac{1}{2}u'(W-qx^*)(-q)+\frac{1}{2}u'(W-D+(1-q)x^*)(1-q)=0.$$

Differentiating this w.r.t. W yields

$$\begin{aligned}u''(W-qx^*)(-q)+u''(W-qx^*)(-q)^2\frac{dx^*}{dW}+u''(W-D+(1-q)x^*)(1-q)\\ +u''(W-D+(1-q)x^*)(1-q)^2\frac{dx^*}{dW}=0\\ \Longrightarrow \frac{dx^*}{dW}=\frac{-u''(W-qx^*)(-q)-u''(W-D+(1-q)x^*)(1-q)}{u''(W-qx^*)(-q)^2+u''(W-D+(1-q)x^*)(1-q)^2}.\end{aligned}$$

Note that $x^* < D$ and DARA means

$$W-D+(1-q)x^* = W-qx^*+(x^*-D) < W-qx^* \implies R_a(W-D+(1-q)x^*) > R_a(W-qx^*).$$

Therefore, the numerator is:

$$\begin{aligned}&-u''(W-qx^*)(-q)-u''(W-D+(1-q)x^*)(1-q)\\ &= -\frac{u''(W-qx^*)}{u'(W-qx^*)}u'(W-qx^*)(-q)-\frac{u''(W-D+(1-q)x^*)}{u'(W-D+(1-q)x^*)}u'(W-D+(1-q)x^*)(1-q)\\ &= R_a(W-qx^*)u'(W-qx^*)(-q)+R_a(W-D+(1-q)x^*)u'(W-D+(1-q)x^*)(1-q)\\ &> R_a(W-qx^*)u'(W-qx^*)(-q)+R_a(W-qx^*)u'(W-D+(1-q)x^*)(1-q)\\ &= R_a(W-qx^*)\underbrace{\Big(u'(W-qx^*)(-q)+u'(W-D+(1-q)x^*)(1-q)\Big)}_{=0 \text{ by FOC}}=0.\end{aligned}$$

Thus, we have $\frac{dx^*}{dW}=\frac{(+)}{(-)}<0$.

3. Consider an investor whose utility function over money is

$$u(w)=2w^{\frac{1}{2}}.$$

The investor can invest in a riskless asset that returns 1 (gross return per ¥1 invested) for sure, or a risky asset that returns 1.4 with probability $\frac{3}{4}$ and 0.8 with probability $\frac{1}{4}$.

(a) Suppose the investor's initial wealth is ¥1000. Letting x denote the amount invested in the risky asset, write the investor's expected utility as a function of x.

Solution: We have

$$g(x) = \begin{cases} 1000 - x + 1.4x = 1000 + 0.4x & \text{with probability } \frac{3}{4} \\ 1000 - x + 0.8x = 1000 - 0.2x & \text{with probability } \frac{1}{4} \end{cases}$$

So,

$$\begin{aligned} U(g(x)) &= \frac{3}{4}u(1000+0.4x) + \frac{1}{4}u(1000-0.2x) \\ &= \frac{3}{4}\left(2(1000+0.4x)^{\frac{1}{2}}\right) + \frac{1}{4}\left(2(1000-0.2x)^{\frac{1}{2}}\right) \end{aligned}$$

(b) Find the optimal amount to invest in the risky asset.

Solution: Investor solves:

$$\max_x \frac{3}{4}\left(2(1000+0.4x)^{\frac{1}{2}}\right) + \frac{1}{4}\left(2(1000-0.2x)^{\frac{1}{2}}\right)$$

FOC is given by

$$\left(\frac{3}{4}\right)\left(\frac{4}{10}\right)(1000+0.4x)^{-\frac{1}{2}} - \left(\frac{1}{4}\right)\left(\frac{2}{10}\right)(1000-0.2x)^{-\frac{1}{2}} = 0$$

$$\begin{aligned} \Rightarrow \left(\frac{3}{10}\right)(1000+0.4x)^{-\frac{1}{2}} &= \left(\frac{1}{20}\right)(1000-0.2x)^{-\frac{1}{2}} \\ \Rightarrow \left(\frac{10}{3}\right)^2(1000+0.4x) &= (20)^2(1000-0.2x) \\ \Rightarrow x &= \frac{(400)(1000) - \left(\frac{100}{9}\right)(1000)}{\left(\frac{100}{9}\right)(0.4) + (400)(0.2)} \approx 4605.263. \end{aligned}$$

Note that $x^* = 4605.263$ is greater than the initial wealth. So, if so-called "short sale" is possible, the investor will borrow additional \$3,605.263 to invest in the risky asset. If borrowing is not possible, then the investor will put the maximum amount in the risky asset. That is, $x^* = 1000$.

4. An investor with initial wealth w_0 is trying to allocate her wealth between a safe asset with constant return $R > 0$ and a risky asset with random return $\boldsymbol{z}$, where $\boldsymbol{z}$ has distribution function F and $\mathrm{E}[\boldsymbol{z}] > R$. Letting x be the *proportion* of wealth invested in the risky asset ($0 \le x \le 1$), her wealth will be:

$$w = \big((1-x)R + xz\big)w_0 = \big(R + x(z-R)\big)w_0,$$

where z is the realized return. Suppose the investor's utility over wealth, $u(\cdot)$, exhibits constant relative risk aversion. Show that the optimal proportion of wealth invested in the risky asset is independent of her initial wealth. That is, show that $\frac{dx^*}{dw_0} = 0$.

Solution: The investor's expected utility maximization is

$$\max_x \mathrm{E}\big[u\big(w(x)\big)\big] \iff \max_x \int u\big((R+x(z-R))w_0\big)\,dF(z).$$

The first order condition is

$$\begin{aligned} & \int u'\big((R+x^*(z-R))w\big)(z-R)w\,dF(z) \;=\; 0 \\ \iff \quad & \int u'\big((R+x^*(z-R))w\big)(z-R)\,dF(z) \;=\; 0 \quad (*) \end{aligned}$$

Differentiating the FOC $(*)$ with respect to w yields (note that we use chain rule on x^*)

$$\begin{aligned} & \int u''\big((R+x^*(z-R))w\big)(z-R)^2 w \frac{dx^*}{dw}\,dF(z) \\ & \qquad + \int u''\big((R+x^*(z-R))w\big)(z-R)(R+x^*(z-R))\,dF(z) \;=\; 0 \end{aligned}$$

$$\Rightarrow \quad \frac{dx^*}{dw} \;=\; \frac{-\int u''\big((R+x^*(z-R))w\big)(z-R)(R+x^*(z-R))\,dF(z)}{\int u''\big((R+x^*(z-R))w\big)(z-R)^2 w\,dF(z)}$$

Looking at the numerator, we see that

$$\begin{aligned} & \int u''\big((R+x^*(z-R))w\big)(z-R)(R+x^*(z-R))\,dF(z) \\ = \;& \int \frac{wu''\big((R+x^*(z-R))w\big)(R+x^*(z-R))}{u'\big((R+x^*(z-R))w\big)} \left(\frac{u'\big((R+x^*(z-R))w\big)(z-R)}{w}\right) dF(z) \\ = \;& \frac{\text{constant}}{w} \int u'\big((R+x^*(z-R))w\big)(z-R)\,dF(z) \quad \text{by CRRA} \\ = \;& 0 \quad \text{by } (*). \end{aligned}$$

So, $\frac{dx^*}{dw} = 0$, as required.

5. Consider a decision maker who faces following two gambles

- Gamble A: Roll a fair, six-sided die. Get paid according to the number on top.
- Gamble B: Roll a fair, six-sided die until an even number comes up on top. Get paid according to the number.

Which gamble will the decision maker prefer? What does it depend on?

Solution: As seen in the graph below, F_B first order stochastically dominate F_A, so any decision maker with non-decreasing $u(\cdot)$ will prefer B over A.