Advanced Microeconomics I

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Problem Set 3: Suggested Solutions

1. There are I individuals, and individual i's utility function is given by

$$u_i(x_1,x_2) = a_i x_1 + b_i x_2^{c_i}, \quad \text{where } a_i, b_i > 0 \text{ and } 0 < c_i < 1.$$

(a) Show that the individuals' indirect utility functions have the Gorman form. You may assume that the solution to the utility maximization problem will be interior and that the second order condition is satisfied.

Solution: Solving the first order condition yields,

$$MRS = \frac{a_i}{b_i c_i x_2^{c_i - 1}} = \frac{p_1}{p_2} \iff x_2 = \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}}.$$

Substituting this into the budget constraint $p_1 x_1 + p_2 x_2 = w_i$ yields,

$$p_1 x_1 + p_2 \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}} = w_i \implies x_1 = \frac{w_i - p_2 \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}}}{p_1}.$$

Thus, the indirect utility function is

$$\begin{aligned} v(p,w_i) &= a_i \left(\frac{w_i - p_2 \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}}}{p_1} \right) + \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{c_i}{1-c_i}} \\ &= \left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{c_i}{1-c_i}} - \frac{a_i p_2}{p_1}\left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}} + \frac{a_i w_i}{p_1}, \end{aligned}$$

which is a linear function of w_i and thus in Gorman form.

(b) For what values of a_i, b_i, and c_i will the aggregate demand be a function of price and aggregate wealth only?

Solution: For the aggregate demand function to depend only on prices and the aggregate wealth, and not the distribution of the aggregate wealth, we need every individual to have an indirect utility function that is in Gorman form (linear function of wealth) with the same slope. Note however that this doe snot require that $a_i = a_j$ for i and j. To see this note that we can transform each individual's utility function by dividing by $a_i > 0$, which preserves the individual's preference. Then, individual i's utility function is now

$$\tilde{u}_i(x_1,x_2) = \frac{u_i(x_1,x_2)}{a_i} = x_1 + \frac{b_i}{a_o} x_2^{c_i}, \quad \text{where } a_i, b_i > 0 \text{ and } 0 < c_i < 1.$$

The corresponding indirect utility function is

$$\tilde{v}(p,w_i) = \frac{v(p,w_i)}{a_i} = \frac{1}{a_i}\left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{c_i}{1-c_i}} - \frac{p_2}{p_1}\left(\frac{b_i c_i p_1}{a_i p_2}\right)^{\frac{1}{1-c_i}} + \frac{w_i}{p_1}.$$

Thus, everyone has the same slope $\frac{1}{p_1}$.

For the remainder of the question, assume that there are only two consumers ($I = 2$), and $a_1 = 1$, $a_2 = 2$, $b_1 = b_2 = 1$, and $c_1 = c_2 = \frac{1}{2}$. Suppose the social welfare function for the economy is given by $W(u_1,u_2) = \min\{u_1,u_2\}$.

(c) Determine the circumstances under which there will be a representative consumer who can be used for aggregate welfare analysis.

Solution: A representative consumer exists when wealth assignment maximizes the Social welfare function. That is, if it solves:

$$\max_{w_1,w_2} W(v_1(p,w_1), v_2(p,w_2)) \text{ s.t. } w_1 + w_2 = w. \iff \max_{w_1} \{(v_1(p,w_1),\ v_2(p,w-w_1)\}.$$

From part (a), we have,

$$v_1(p,w_1) = \frac{p_1}{2p_2} - \frac{p_2}{p_1}\left(\frac{p_1}{2p_2}\right)^2 + \frac{w_1}{p_1} = \frac{p_1}{2p_2} - \frac{p_1}{4p_2} + \frac{w_1}{p_1} = \frac{p_1}{4p_2} + \frac{w_1}{p_1}$$

$$v_2(p,w_2) = \frac{p_1}{4p_2} - \frac{2p_2}{p_1}\left(\frac{p_1}{4p_2}\right)^2 + \frac{2w_2}{p_1} = \frac{p_1}{4p_2} - \frac{p_1}{8p_2} + \frac{2w_2}{p_1} = \frac{p_1}{8p_2} + \frac{2w_2}{p_1}.$$

The solution to the maximization occurs where $v_1 = v_2$. Thus,

$$\frac{p_1}{4p_2} + \frac{w_1}{p_1} = \frac{p_1}{8p_2} + \frac{2(w-w_1)}{p_1} \iff \frac{p_1^2}{4p_2} + w_1 = \frac{p_1^2}{8p_2} + 2w - 2w_2$$

$$\iff w_1^* = \frac{2w}{3} - \frac{p_1^2}{24p_2} \implies w_2^* = \frac{w}{3} + \frac{p_1^2}{24p_2}$$

(d) Find the indirect utility function for the representative consumer.

Solution:

$$V(p,w) = \min\{v_1(p,w_1^*),\ v_2(p_2,w_2^*)\} = \frac{p_1}{4p_2} + \frac{1}{p_1}\left(\frac{2w}{3} - \frac{p_1^2}{24p_2}\right)$$

$$= \frac{2w}{3p_1} + \frac{p_1}{4p_2} - \frac{p_1}{24p_2} = \frac{5}{24p_2} + \frac{2w}{3p_1}.$$

2. Consider a profit maximizing firm whose single-output technology is given by the production function $f : \mathbb{R}_+^L \to \mathbb{R}_+$ satisfying the usual assumptions (continuity, quasiconcavity, and monotonicity). Let $w \gg 0$ and $p > 0$ denote the input and the output prices, respectively. In the following, assume that the solution to the profit maximization problem is unique.

(a) Show that the firm's profit function, $\pi(w,p)$, is a convex function of (w,p).

Solution: Let $y(w,p)$ and $z(w,p)$ be the supply and the unconditional demand functions. Then,

$$\begin{aligned}\pi\left(\alpha(w,p)+(1-\alpha)(w',p')\right) &= \left(\alpha p+(1-\alpha)p'\right)y(w,p)-\left(\alpha w+(1-\alpha)w'\right)\cdot z(w,p)\\ &= \alpha\left(py(w,p)-w\cdot z(w,p)\right)+(1-\alpha)\left(p'y(w,p)-w'\cdot z(w,p)\right)\\ &\leq \alpha\pi(w,p)+(1-\alpha)\pi(w',p'),\ \text{as required.}\end{aligned}$$

(b) Show whether the firm's supply function $y(w,p)$ and the unconditional demand function $z(w,p)$ satisfy the law of supply and the law of demand, respectively. For this part, assume that $\pi(w,p)$, $y(w,p)$ and $z(w,p)$ are differentiable.

Solution: By part (a), $\pi(w,p)$ is convex, which means $D_{(w,p)}\pi(w,p)$ is positive semidefinite. This, together with Hotelling's lemma yields,

$$\frac{\partial y(w,p)}{\partial p}=\frac{\partial^2\pi(w,p)}{\partial p^2}\geq 0 \quad\text{and}\quad \frac{\partial z_\ell(w,p)}{\partial w_\ell}=-\frac{\partial^2\pi(w,p)}{\partial w_\ell^2}\leq 0,$$

which are laws of supply and demand, respectively.

(c) Show again whether the firm's supply function and the unconditional demand function satisfy the law of supply and the law of demand, respectively. For this part, do NOT assume that $\pi(w,p)$, $y(w,p)$ and $z(w,p)$ are differentiable.

solution: Let $y=y(w,p)$, $z=z(w,p)$, $y'=y(w',p')$, and $z'=z(w',p')$. Then by the definition of profit maximization,

$$\begin{aligned}&(p,w)\cdot\left((y,-z)-(y'-z')\right)\ +\ (p',w')\cdot\left((y',-z')-(y-z)\right)\ \geq\ 0\\ &\implies \left((p,w)-(p',w')\right)\cdot\left((y,-z)-(y'-z')\right)\ \geq\ 0.\end{aligned}$$

The first component of the above (vector) inequality and the $\ell+1$-th component yields the law of supply and demand, respectively:

$$\begin{aligned}&\left(p-p'\right)\left(y-y'\right)\geq 0\\ &\left(w_\ell-w'_\ell\right)\left(-z_\ell-(-z'_\ell)\right)\geq 0 \implies \left(w_\ell-w'_\ell\right)\left(z_\ell-z'_\ell\right)\leq 0.\end{aligned}$$

(d) Suppose there are J firms, and each firm's supply function satisfies the law of supply and the input demand function satisfies the law of demand. Show whether the aggregate supply and the aggregate demand satisfy the law of supply and demand, respectively.

Solution: Suppose $p>p'$. Then $y_j(w,p)\geq y_j(w,p')$ for all j by the law of supply. Thus, $\sum_j y_j(w,p)\geq\sum_j y_j(w,p')$. Suppose $w_\ell>w'_\ell$ and $w_k=w'_k$ for all $k\neq\ell$. Then $z_{\ell j}(w,p)\leq z_{\ell j}(w',p)$ for all j by the law of demand. Thus, $\sum_j z_{\ell j}(w,p)\leq z_{\ell j}(w',p)$. Thus, the aggregate supply and the aggregate demand satisfies the laws of supply and demand, respectively.

3. A firm uses a production function $f(z) = \min\ \{z_1^a,\ z_2^b\}$, $z \in \mathbb{R}_+^L$, to produce an output.

(a) Find the firm's cost function. Determine the conditions under which the cost function will be concave, linear, and convex function of the output level, y, respectively (you may use the second derivative condition).

Solution: For Leontief production function, cost-minimizing bundle occurs at the corner point: $z_1^a = z_2^b \implies z_2 = z_1^{\frac{a}{b}}$. Thus,

$$\min\left\{z_1^a,\ \left(z_1^{\frac{a}{b}}\right)^b\right\} = y \implies z_1(w,y) = y^{\frac{1}{a}} \implies z_2(w,y) = y^{\frac{1}{b}} \implies c(w,y) = w_1 y^{\frac{1}{a}} + w_2 y^{\frac{1}{b}}$$

$$\implies \frac{\partial^2 c(w,y)}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{w_1}{a} y^{\frac{1}{a}-1} + \frac{w_2}{b} y^{\frac{1}{b}-1}\right) = \frac{w_1}{a}\left(\frac{1}{a}-1\right) y^{\frac{1}{a}-2} + \frac{w_2}{b}\left(\frac{1}{b}-1\right) y^{\frac{1}{b}-1}.$$

Since the second derivative condition for concavity/convexity has to hold for all $w_1 > 0$ and $w_2 > 0$, we have

$$c(w,y) \text{ is } \begin{cases} \text{concave} & \text{if } \frac{1}{a} \le 1 \text{ and } \frac{1}{b} \le 1, \text{ or equivalently } a \ge 1 \text{ and } b \ge 1 \\ \text{linear} & \text{if } a = b = 1 \\ \text{convex} & \text{if } a \le 1 \text{ and } b \le 1. \end{cases}$$

(b) Assuming that the cost function is convex and that $a = b$, find the firm's profit function.

Solution:

$$p = c'(w,y) \implies p = \frac{w_1}{a} y^{\frac{1}{a}-1} + \frac{w_2}{a} y^{\frac{1}{a}-1} \implies p = \frac{w_1 + w_2}{a} y^{\frac{1-a}{a}} \implies y = \left(\frac{ap}{w_1 + w_2}\right)^{\frac{a}{1-a}}.$$

$$\pi(p,w) = p\left(\frac{ap}{w_1 + w_2}\right)^{\frac{a}{1-a}} - (w_1 + w_2)\left(\frac{ap}{w_1 + w_2}\right)^{\frac{1}{1-a}}$$

(c) Assuming that the cost function is linear and that $a = b$, find the firm's profit function.

Solution: Linear cost function means that $a = b = 1$. Thus,

$$\text{marginal profit } = p - c'(w,y) = p - (w_1 + w_2).$$

Therefore,

$$y(p,w) = \begin{cases} 0 & \text{if } p < w_1 + w_2 \\ [0,\infty) & \text{if } p = w_1 + w_2 \\ \text{undefined} & \text{if } p > w_1 + w_2 \end{cases}$$

$$\implies \pi(p,w) = \begin{cases} 0 & \text{if } p \le w_1 + w_2 \\ \text{undefined} & \text{if } p > w_1 + w_2. \end{cases}$$

4. A firm has two factories that produce (the same) output good using labor as the only input. The two factories have the same production function, $f_1(z_1) = z_1^{\frac{1}{a}}$ and $f_2(z_2) = z_2^{\frac{1}{a}}$, where $a > 0$ and z_j is the labor input in factory j. The first factory is located in province 1 where wage is w_1 while the second one is located in province 2, where wage is w_2.

(a) Find the firm's cost function.

Solution: Since there is only one input good, the conditional input demand is given by the inverse of the production function:

$$\begin{aligned}
f_1(z_1) = z_1^{\frac{1}{a}} = y_1 &\implies z_1 = y_1^a \implies c_1(w_1, y_1) = w_1 y_1^a \\
f_2(z_2) = z_2^{\frac{1}{2}} = y_2 &\implies z_2 = y_2^a \implies c_2(w_2, y_2) = w_2 y_2^a.
\end{aligned}$$

The overall cost function for producing y units of output is found by optimally allocating production across the two factories.

$$\min_{0 \leq y_1 \leq y} c_1(w_1, y_1) + c_2(w_2, y - y_1) \iff \min_{0 \leq y_1 \leq y} w_1 y_1^a + w_2 (y - y_1)^a.$$

Let $g(y_1) = w_1 y_1^a + w_2(y - y_1)^a$. Note that the (sufficient) second order condition for minimization, $g''(y_1) = a(a-1) w_1 y_1^{a-2} + a(a-1) w_2 (y - y_1)^{a-2} > 0$, is guaranteed only if $a > 1$. So, assume $a > 1$. Then,

$$\begin{aligned}
a w_1 y_1^{a-1} - a w_2 (y - y_1)^{a-1} = 0 &\iff y_1^{a-1} = \frac{w_2}{w_1} (y - y_1)^{a-1} \\
y_1 = \left(\frac{w_2}{w_1} \right)^{\frac{1}{a-1}} (y - y_1) &\iff w_1^{\frac{1}{a-1}} y_1 + w_2^{\frac{1}{a-1}} = w_2^{\frac{1}{a-1}} y \\
y_1 = \left(\frac{w_2^{\frac{1}{a-1}}}{w_1^{\frac{1}{a-1}} + w_2^{\frac{1}{a-1}}} \right) y &\implies y_2 = \left(\frac{w_1^{\frac{1}{a-1}}}{w_1^{\frac{1}{a-1}} + w_2^{\frac{1}{a-1}}} \right) y \\
\implies c(w, y) &= w_1 \left(\frac{w_2^{\frac{1}{a-1}}}{w_1^{\frac{1}{a-1}} + w_2^{\frac{1}{a-1}}} \right)^a y^a + w_2 \left(\frac{w_1^{\frac{1}{a-1}}}{w_1^{\frac{1}{a-1}} + w_2^{\frac{1}{a-1}}} \right)^a y^a \\
&= \left(\frac{w_1 w_2^{\frac{1}{a-1}} + w_2 w_1^{\frac{1}{a-1}}}{w_1^{\frac{1}{a-1}} + w_2^{\frac{1}{a-1}}} \right)^a y^a.
\end{aligned}$$

When, $a \leq 1$, the objective function g is linear or concave. So, the cost minimizing y_1 will occur at the boundary $y_1 = 0$ or $y_1 = y$. Thus,

$$c(w, y) = \begin{cases}
\left(\frac{w_1 w_2^{\frac{1}{a-1}} + w_2 w_1^{\frac{1}{a-1}}}{w_1^{\frac{1}{a-1}} + w_2^{\frac{1}{a-1}}} \right)^a y^a & \text{if } a > 1 \\
w_1 y^a & \text{if } a \leq 1 \text{ and } w_1 \leq w_2 \\
w_2 y^a & \text{if } a \leq 1 \text{ and } w_2 \leq w_2.
\end{cases}$$

(b) Interpret this result.

solution: The firm exhibits decreasing returns to scale when $a > 1$, constant returns to scale when $a = 0$, and increasing returns to scale when

$0 < a < 1$. The firm uses both factories only when they have decreasing returns to scale. If the factories exhibits constant or increasing returns to scale, it is better off making all of its production in one factory with the lower input cost.

5. **(OPTIONAL - does not need to be handed in)** LetsBurnInvestorsMoney.com is a startup firm that produces a product called *pipedream*. The firm acts as a price taker; however, the firm has decided that growth is more important than profitability. Therefore, the firm has chosen to maximize output instead of profit. What the firm can spend on input goods is constrained by how much cash it has on hand.

   Suppose your careful observation of the firm's behavior has revealed that when the input prices are $w = (w_1, w_2)$ and the firm has \$$C$ cash on hand, the firm's output is:

$$y(w, C) = \left(\frac{1}{3w_1}\right)^{\frac{1}{3}} \left(\frac{2}{3w_2}\right)^{\frac{2}{3}} C.$$

   Find the firm's input demand.

   Solution: This firm solves

$$\max_{z} \; f(z) \quad \text{s.t.} \quad w \cdot z = C.$$

   The Lagrangian is

$$\mathscr{L} = f(z) + \lambda[C - w \cdot z].$$

   The value function is

$$y(w, C) = \left(\frac{1}{3w_1}\right)^{\frac{1}{3}} \left(\frac{2}{3w_2}\right)^{\frac{2}{3}} C.$$

   By the envelope theorem (compare with Roy's Identity),

$$\frac{\partial y}{\partial w_i} = \frac{\mathscr{L}}{\partial w_i} = -\lambda z_i \quad \text{and} \quad \frac{\partial y}{\partial C} = \frac{\mathscr{L}}{\partial C} = \lambda.$$

   Therefore, the firm's input demands are

$$z_1(w, C) = -\frac{\frac{\partial y}{\partial w_1}}{\frac{\partial y}{\partial C}} = -\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3w_1}\right)^{-\frac{2}{3}}\left(-\frac{1}{3w_1^2}\right)\left(\frac{2}{3w_2}\right)^{\frac{2}{3}} C}{\left(\frac{1}{3w_1}\right)^{\frac{1}{3}}\left(\frac{2}{3w_2}\right)^{\frac{2}{3}}} = \frac{C}{3w_1}$$

$$z_2(w, C) = -\frac{\frac{\partial y}{\partial w_2}}{\frac{\partial y}{\partial C}} = -\frac{\left(\frac{2}{3}\right)\left(\frac{1}{3w_1}\right)^{\frac{1}{3}}\left(\frac{2}{3w_2}\right)^{-\frac{1}{3}}\left(-\frac{2}{3w_2^2}\right) C}{\left(\frac{1}{3w_1}\right)^{\frac{1}{3}}\left(\frac{2}{3w_2}\right)^{\frac{2}{3}}} = \frac{2C}{3w_2}.$$