# Advanced Microeconomics I

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Problem Set 2: Suggested Solutions

1. Consider a consumer whose utility function over food and clothing is $u(x_f, x_c)$. The consumer has income $w$ and faces prices $p = (p_f, p_c)$. Suppose further that she lives in a country where clothing is taxed but food is not. Tax on clothing is *ad valorem* so that the effective price consumer pays on clothing is $(1+t)p_c$, where $t$ is the tax rate.

   (a) Formulate the consumer's utility maximization problem.
   **Solution:** The consumer solves

$$\max_{x_f, x_c} u(x_f, x_c) \quad \text{s.t.} \quad p_f x_f + (1+t)p_c x_c = w.$$

   (b) Suppose the consumer's indirect utility function is given by

$$v(p,t,w) = \frac{3w^2}{16(1+t)p_f p_c}.$$

   Derive the Marshallian demands for clothing and food, $x_c(p,t,w)$ and $x_f(p,t,w)$. How do they change as $t$ changes?
   **Solution:** The Lagrangian for the optimization problem is

$$\mathscr{L} \ = \ u(x_f, x_c) + \lambda \left[ w - p_f x_f - (1+t)p_c x_c \right].$$

   The envelope theorem implies that

$$\frac{\partial \mathscr{L}}{\partial p_c} = -\lambda^*(1+t)x_c^*, \qquad \frac{\partial \mathscr{L}}{\partial p_f} = -\lambda^* x_f^*, \qquad \text{and} \qquad \frac{\partial \mathscr{L}}{\partial w} = \lambda^*$$

$$\Longrightarrow x_f^* = -\frac{\frac{\partial \mathscr{L}}{\partial p_f}}{\frac{\partial \mathscr{L}}{\partial w}} = -\frac{\frac{\partial v(p,t,w)}{\partial p_f}}{\frac{\partial v(p,t,w)}{\partial w}} = -\frac{\frac{-3w^2}{16(1+t)p_f^2 p_c}}{\frac{6w}{16(1+t)p_f p_c}} = \frac{w}{2p_f}$$

$$\Longrightarrow x_c^* = -\frac{1}{1+t}\left(\frac{\frac{\partial \mathscr{L}}{\partial p_c}}{\frac{\partial \mathscr{L}}{\partial w}}\right) = -\frac{1}{1+t}\left(\frac{\frac{\partial v(p,t,w)}{\partial p_c}}{\frac{\partial v(p,t,w)}{\partial w}}\right) = -\frac{1}{1+t}\left(\frac{\frac{-3w^2}{16(1+t)p_f p_c^2}}{\frac{6w}{16(1+t)p_f p_c}}\right) = \frac{1}{(1+t)}\left(\frac{w}{2p_c}\right).$$

   Thus, increase in $t$ has no effect on the demand for food while it decreases the demand for clothing.

(c) Derive the expenditure function, $e(p,t,u)$ and the Hicksian demands. How do they change as $t$ changes? Interpret these results.

**Solution:** We have

$$v(p,t,e(p,t,u)) = u \implies \frac{3e(p,t,u)^2}{16(1+t)p_f p_c} = u \implies e(p,t,u) = \left(\frac{16(1+t)p_f p_c u}{3}\right)^{\frac{1}{2}}.$$

Thus, expenditure is increasing in $t$ (differentiate with respect to $t$ to find by how much). That is, as the tax increases, the individual need to spend more money to achieve the same level of utility as before.

To find the Hicksian demands, note that the expenditure minimization problem and the associated Lagrangian are

$$\min_{x_f,x_c} p_f x_f + (1+t)p_c x_c \quad \text{s.t.} \quad u(x_f,x_c) \geq u$$
$$\implies \mathcal{L} = p_f x_f + (1+t)p_c x_c - \mu\left[u - u(x_f,x_c)\right].$$

Thus, using the envelope, we obtain

$$\implies h_f(p,t,u) = \frac{\partial e(p,t,u)}{\partial p_f} = \frac{1}{2}\left(\frac{16(1+t)p_f p_c u}{3}\right)^{-\frac{1}{2}}\left(\frac{16(1+t)p_c u}{3}\right) = \frac{1}{2}\left(\frac{16(1+t)p_c u}{3p_f}\right)^{\frac{1}{2}}$$

$$h_c(p,t,u) = \left(\frac{1}{1+t}\right)\frac{\partial e(p,t,u)}{\partial p_c} = \left(\frac{1}{1+t}\right)\left(\frac{1}{2}\right)\left(\frac{16(1+t)p_f u}{3p_c}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{16 p_f u}{3(1+t)p_c}\right)^{\frac{1}{2}}.$$

Thus, the Hicksian demand for clothing is decreasing in $t$ while the Hicksian demand for food is increasing (differentiate with respect to $t$ to find by how much). Increase in $t$, the tax on clothing, makes clothing more expensive relative to food. Thus, the individual responds by substituting away from clothing toward food.

2. A consumer's expenditure function is given by

$$e(p,u) = \left(p_1^{-2} + p_2^{-2}\right)^{-\frac{1}{2}} u.$$

**Remark:** This is a revised solution correcting the error in the question. Exponents were missing the negative signs in the original question. In case you are wondering, this is an expenditure function for a CES utility function, $u(x_1,x_2) = \left(x_1^{\frac{2}{3}} + x_2^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

(a) Find the Slutsky matrix.

**Solution:** To find the Slutsky matrix, we first need the Marshallian demand, which can be derived from the indirect utility function using Roy's identity. Using $e(p, v(p,w) = w$, we obtain,

$$e(p,v(p,w)) = \left(p_1^{-2} + p_2^{-2}\right)^{-\frac{1}{2}} v(p,w) = w \quad \implies \quad v(p,w) = \left(p_1^{-2} + p_2^{-2}\right)^{\frac{1}{2}} w.$$

Roy's identity then yields,

$$x_\ell(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_\ell}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\frac{1}{2}\left(p_1^{-2}+p_2^{-2}\right)^{-\frac{1}{2}}(-2)p_\ell^{-3}w}{\left(p_1^{-2}+p_2^{-2}\right)^{\frac{1}{2}}} = \left(\frac{p_\ell^{-3}}{p_1^{-2}+p_2^{-2}}\right)w.$$

Therefore, letting $\ell \neq k$, we have

$$\begin{aligned}
\frac{\partial x_\ell}{\partial p_\ell} + x_\ell \frac{\partial x_\ell}{\partial w} &= \left(\frac{-3p_\ell^{-4}(p_1^{-2}+p_2^{-2}) - p_\ell^{-3}(-2p_\ell^{-3})}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w + \left(\frac{p_\ell^{-3}}{p_1^{-2}+p_2^{-2}}\right)w\left(\frac{p_\ell^{-3}}{p_1^{-2}+p_2^{-2}}\right) \\
&= \left(\frac{-3p_\ell^{-6} - 3p_\ell^{-4}p_k^{-2} + 2p_\ell^{-6}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w + \left(\frac{p_\ell^{-6}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w \\
&= \left(\frac{-p_\ell^{-6} - 3p_\ell^{-4}p_k^{-2}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w + \left(\frac{p_\ell^{-6}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w = \left(\frac{-3p_\ell^{-4}p_k^{-2}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w \\
\frac{\partial x_\ell}{\partial p_k} + x_k \frac{\partial x_\ell}{\partial w} &= \left(\frac{-p_\ell^{-3}(-2p_k^{-3})}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w + \left(\frac{p_k^{-3}}{p_1^{-2}+p_2^{-2}}\right)w\left(\frac{p_\ell^{-3}}{p_1^{-2}+p_2^{-2}}\right) \\
&= \underbrace{\left(\frac{2p_\ell^{-3}p_k^{-3}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w}_{(*)} + \underbrace{\left(\frac{p_\ell^{-3}p_k^{-3}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w}_{(**)} = \left(\frac{3p_\ell^{-3}p_k^{-3}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w \\
\implies S(p,w) &= \begin{bmatrix} \frac{\partial x_1}{\partial p_1} + x_1\frac{\partial x_1}{\partial w} & \frac{\partial x_1}{\partial p_2} + x_2\frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial p_1} + x_1\frac{\partial x_2}{\partial w} & \frac{\partial x_2}{\partial p_2} + x_2\frac{\partial x_2}{\partial w} \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{-3p_1^{-4}p_2^{-2}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w & \left(\frac{3p_1^{-3}p_2^{-3}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w \\ \left(\frac{3p_2^{-3}p_1^{-3}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w & \left(\frac{-3p_2^{-4}p_1^{-2}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w \end{bmatrix}.
\end{aligned}$$

(b) Find the substitution effect, the income effect, and total effect, on good 1 arising from the change in the price of good 2.

**Solution:** Recall that $D_p h(p,u) = S(p,w)$, so the entries of the Slutsky matrix are the substitution effects. Thus, the substitution effect is

$$S_{12} = \left(\frac{3p_1^{-3}p_2^{-3}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w.$$

Note that this is positive since if the price of good 2 increases, substitution effect should increase the demand for good 1. The income effect (using $(**)$ from part (a)) is

$$-x_2(p,w)\frac{\partial x_1(p,w)}{\partial w} = -\left(\frac{p_1^{-3}p_2^{-3}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w,$$

and the total effect (using $(*)$ from part (a)) is

$$\frac{\partial x_1(p,w)}{\partial p_2} = \left(\frac{2p_1^{-3}p_2^{-3}}{\left(p_1^{-2}+p_2^{-2}\right)^2}\right)w.$$

3. Ellsworth's utility function is $u(x_1,x_2) = \min\{x_1,x_2\}$. Ellsworth has ¥150 and the price of the two goods are both ¥1. Ellsworth's boss is thinking of sending him to another town where the price of good 1 is ¥1 and the price of good 2 is ¥2. The boss offers no raise in pay. Ellsworth, who understands compensating and equivalent variation perfectly, complains bitterly. He says that although he doesn't mind moving for its own sake and the new town is just as pleasant as the old, having to move is as bad as a cut in pay of ¥$A$. He also says that he wouldn't mind moving if when he moved he got a raise of ¥$B$. What are $A$ and $B$ equal to?

**Solution:** For this utility function, the utility maximizing consumption bundle occurs where the "kink" in the indifference curve is tangent to the budget line. That is, where

$$x_1 = x_2 \quad \text{and} \quad p_1x_1 + p_2x_2 = w.$$

Solving these two equations yields the demand and the indirect utility functions:

$$x(p,w) = \left(\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2}\right) \quad \text{and} \quad v(p,w) = \frac{w}{p_1+p_2}.$$

Letting $p^0 = (1,1)$ and $p^1 = (1,2)$, we have

$$v(p^0,w) = \frac{150}{1+1} = 75 \quad \text{and} \quad v(p^1,w) = \frac{150}{1+2} = 50.$$

The equivalent variation, $A$, can be found by

$$v(p^0, w+EV) = v(p^1,w) \implies \frac{150+EV}{1+1} = 50 \implies EV = 100 - 150 = -50.$$

So, the move would be equivalent to ¥50 pay cut at the old prices. The compensating variation, $B$, can be found by

$$v(p^1, w-CV) = v(p^0,w) \implies \frac{150-CV}{1+2} = 75 \implies CV = 150 - 225 = -75.$$

So, if he moves, he would need to be compensated ¥75 to make him equally as well as before.

4. In the following, let $EV(p^0,p^1,w)$ and $EV(p^0,p^2,w)$ denote the equivalent variation measure of welfare change between $(p^0,w)$ and $(p^1,w)$ and between $(p^0,w)$ and $(p^2,w)$, respectively. Let $CV(p^0,p^1,w)$ and $CV(p^0,p^2,w)$ denote the analogous for compensating variation measure of welfare change. Note that we are considering cases in which the individual's wealth does not change.

   (a) Show that the equivalent variation measure gives a correct welfare ranking of $p^1$ versus $p^2$. That is, $EV(p^0,p^1,w) > EV(p^0,p^2,w)$ if and only if $v(p^1,w) > v(p^2,w)$.

**Solution:** We have

$$\begin{aligned}
&EV(p^0,p^1,w) > EV(p^0,p^2,w)\\
\iff &e(p^0,v(p^1,w)) - e(p^0,v(p^0,w)) > e(p^0,v(p^2,w)) - e(p^0,v(p^0,w))\\
\iff &e(p^0,v(p^1,w)) > e(p^0,v(p^2,w))\\
\iff &v(p^1,w) > v(p^2,w) \text{ since } e(p^0,u) \text{ is increasing in } u.
\end{aligned}$$

The steps (b)-(d) below construct an example where $CV(p^0,p^1,w)$ and $CV(p^0,p^1,w)$ do not give a correct welfare ranking of $p^1$ versus $p^2$. First, let $u(x) = x_1 + \phi(x_2)$, where $\phi(\cdot)$ is increasing and strictly concave. This utility function is an example of a quasilinear utility function, which is linear in one of the goods ($x_1$ in this case). In the following, assume that $\phi(\cdot)$ is differentiable and that the Marshallian demand will be interior (i.e., $x(p,w) \gg 0$).

(b) Show that good 1 is a normal good and that the wealth effect on good 2 (the nonlinear part of the quasilinear utility) is zero (i.e., $\frac{\partial x_2(p,w)}{\partial w} = 0$).

**Solution:** Assuming interior solution, the first order condition for the utility maximization problem is the "MRS = price ratio" condition:

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{1}{\phi'(x_2)} = \frac{p_1}{p_2} \implies \phi'(x_2) = \frac{p_2}{p_1}.$$

Since the last equation doesn't involve $x_1$, $x_2(p,w)$ can be found by solving the equation for $x_2$ (see for example, PS1, Q1). Thus, we have

$$x_2(p,w) = (\phi')^{-1}\left(\frac{p_2}{p_1}\right) \implies \frac{\partial x_2(p,w)}{\partial w} = 0 \text{ since } x_2(p,w) \text{ does not depend on } w$$

$$x_1(p,w) = \frac{w - p_2 x_2(p,w)}{p_1} \implies \frac{\partial x_1(p,w)}{\partial w} = \frac{1}{p_1} > 0.$$

(c) Let $p^0 = (p_1^0, p_2^0)$, and obtain $p^1$ from $p^0$ by lowering the price of good 1 slightly. That is, $p^1 = (p_1^1, p_2^0)$ for some $p_1^1 < p_1^0$. Next, obtain $p^2$ from $p^0$ by lowering the price of good 2 until $v(p^1,w) = v(p^2,w)$. That is, $p^2 = (p_1^0, p_2^2)$, where $p_2^2 < p_2^0$ and $v(p^1,w) = v(p^2,w)$. Show that $EV(p^0,p^1,w) = EV(p^0,p^2,w)$, meaning EV ranks $p^1$ and $p^2$ correctly.

**Solution:** Since $v(p^1,w) = v(p^2,w)$ by construction, we have

$$\begin{aligned}
EV(p^0,p^1,w) &= e(p^0,v(p^1,w)) - e(p^0,v(p^0,w))\\
&= e(p^0,v(p^2,w)) - e(p^0,v(p^0,w)) = EV(p^0,p^2,w).
\end{aligned}$$

Note in particular this means that $EV$ correctly determines that the individual likes $p^1$ and $p^2$ equally.

(d) Show that $CV(p^0,p^1,w) < CV(p^0,p^2,w)$, meaning $CV$ ranks $p^1$ and $p^2$ incorrectly.

**Solution:** As shown in the lecture, we have $p_1^1 < p_1^0$ and good 1 is normal. Therefore, $CV(p^0,p^1,w) < EV(p^0,p^1,w)$. Moreover, since the wealth effect on good 2 is zero, $CV(p^0,p^2,w) = EV(p^0,p^2,w)$. Thus, we have

$$CV(p^0,p^1,w) < EV(p^0,p^1,w) = EV(p^0,p^2,w) = CV(p^0,p^2,w).$$

5. Suppose the economy is composed of $N$ households, each differing only in its wealth level and family size. Let $w_i$ and $s_i$ denote household $i$'s wealth and family size, respectively, and let $x_i(p,w_i,s_i)$ and $v_i(p,w_i,s_i)$ be the corresponding Marshallian demand and indirect utility function.

   Show that the aggregate demand only depends on price $p$, aggregate wealth $w = \sum_i w_i$, and average family size $\bar{s} = \frac{\sum_i s_i}{N}$ if every household's indirect utility function is given by:

$$v_i(p,w_i,s_i) = a_i(p) + b(p)w_i + c(p)s_i.$$

   **Solution:** Letting $x_{\ell i}$ denote household $i$'s demand for good $\ell$, we have, by Roy's identity,

$$\begin{aligned} x_{\ell i}(p,w_i,s_i) &= -\frac{\frac{\partial v_i}{\partial p_\ell}}{\frac{\partial v_i}{\partial w_i}} = -\frac{\frac{\partial a_i(p)}{\partial p_\ell} + \frac{\partial b(p)}{\partial p_\ell} w_i + \frac{\partial c(p)}{\partial p_\ell} s_i}{b(p)} \\ \Rightarrow \quad \sum_i x_{\ell i}(p,w_i,s_i) &= \frac{-1}{b(p)} \left( \sum_i \frac{a_i(p)}{p_\ell} + \sum_i \frac{\partial b(p)}{\partial p_\ell} w_i + \sum_i \frac{\partial c(p)}{\partial p_\ell} s_i \right) \\ &= \frac{-1}{b(p)} \left( \sum_i \frac{a_i(p)}{p_\ell} + \frac{\partial b(p)}{\partial p_\ell} w + \frac{\partial c(p)}{\partial p_\ell} N\bar{s} \right). \end{aligned}$$