Advanced Microeconomics I

Fall 2023 - M. Pak

Exercises: Choice under uncertainty and General equilibrium

1 Choice under Uncertainty

1. Consider an individual with initial wealth ¥W and utility function over money given by $u(w) = w^{\frac{1}{2}}$. The individual faces possible loss of ¥L, where L is either W, $\frac{W}{2}$, or 0, each with probability $\frac{1}{3}$. Suppose an insurance that will cover her entire loss is available at price ¥p. That is, for the total price ¥p, the insurance company will make a payment equal to the amount of the realized loss.

   (a) Find the individual's expected utility when she does not buy the insurance.
   Solution:

$$\begin{aligned} U_{no} &= \frac{1}{3}W^{\frac{1}{2}} + \frac{1}{3}\left(\frac{W}{2}\right)^{\frac{1}{2}} + \frac{1}{3}0^{\frac{1}{2}} \\ &= \frac{1+\sqrt{2}}{3\sqrt{2}}W^{\frac{1}{2}} \approx (0.569)W^{\frac{1}{2}}. \end{aligned}$$

   (b) Find the individual's expected utility if she buys the insurance.
   Solution:

$$U_{yes} = (W-p)^{\frac{1}{2}}.$$

   (c) Let $W = 100$. What is the maximum price at which she will buy the insurance?
   Solution: We need

$$\begin{aligned} (W-p)^{\frac{1}{2}} &\geq \frac{1+\sqrt{2}}{3\sqrt{2}}W^{\frac{1}{2}} \\ 100 - \left(\frac{1+\sqrt{2}}{3\sqrt{2}}\right)^2(100) &\geq p \\ p &\leq 100-(0.569)^2(100) = 67.62. \end{aligned}$$

2. Consider an individual whose utility function over money is $u(w) = w^{\frac{1}{2}}$. The individual is facing a risk of losing ¥100 with probability $\frac{1}{2}$ and nothing with probability $\frac{1}{2}$.

(a) Suppose the individual's current wealth is ¥1,000. What is the maximum amount of money that she is willing to pay to avoid this risk?

Solution: The maximum amount of money she is willing to pay, M, must solve

$$\begin{aligned} u(1000-M) &= \frac{1}{2}u(1000-100)+\frac{1}{2}u(1000-0) \\ \sqrt{1000-M} &= \frac{1}{2}\sqrt{900}+\frac{1}{2}\sqrt{1000} \approx 30.81 \\ M &= 1000-949.34 = 50.66 \end{aligned}$$

(b) Suppose the individual's current wealth is ¥10,000. What is the maximum amount of money that she is willing to pay to avoid this risk?

Solution: The maximum amount of money she is now willing to pay, M, must solve

$$\begin{aligned} u(10000-M) &= \frac{1}{2}u(10000-100)+\frac{1}{2}u(10000-0) \\ \sqrt{10000-M} &= \frac{1}{2}\sqrt{9900}+\frac{1}{2}\sqrt{10000} \approx 99.75 \\ M &= 10000-9949.94 = 50.06 \end{aligned}$$

(c) Now, suppose the individual's current wealth is again ¥1000, but now the individual is facing the prospect of gaining ¥100 with probability $\frac{1}{2}$ and nothing with probability $\frac{1}{2}$. What is the maximum amount of money that she is willing to pay to take on this risk?

Solution: Since the individual now pays to take on the risk, the maximum amount of money she is willing to pay, M, must solve

$$\begin{aligned} u(1000) &= \frac{1}{2}u(1100-M)+\frac{1}{2}u(1000-M) \\ 2\sqrt{1000} &= \sqrt{1100-M}+\sqrt{1000-M} \\ 63.25-\sqrt{1000-M} &= \sqrt{1100-M} \\ 63.25^2-2(63.25)\sqrt{1000-M}+1000-M &= 1100-M \\ 3900.56 &= 2(63.25)\sqrt{1000-M} \\ 30.83^2 &= 1000-M \\ M &= 1000-950.76 = 49.24. \end{aligned}$$

That is, the decision maker must be paid 49.24 to not to take on this risk.

(d) How do you interpret these results?

Solution: Parts (a) and (b) shows that an individual's evaluation of uncertainty depends on the initial wealth the individual has. That the absolute value of the amount that needs to be paid in Parts (a) and (c) is different shows that an individual's evaluation of uncertainty involving loss may be different from that involving gain.

3. Consider the insurance example given in the class. Recall that π is the probability that the loss will occur. Suppose the price of insurance is given by $q > \pi$.

   (a) Will a risk averse consumer still fully insure?

   Solution: (Note: This solution uses L for possible loss and x for the amount of insurance purchased). The consumer solves

$$\max_x \pi u(w - qx - L + x) + (1-\pi)u(w - qx).$$

   The first order condition is

$$\pi u'(w - qx^* - L + x^*)(1-q) + (1-\pi)u'(w - qx^*)(-q) = 0$$

$$\Rightarrow \quad u'(w - px^* - L + x^*) = \frac{q(1-\pi)}{\pi(1-q)} u'(w - px^*).$$

   Since $q > \pi$, we have $\frac{q(1-\pi)}{\pi(1-q)} > 1$. So,

$$u'(w - qx^* - L + x^*) > u'(w - qx^*).$$

   If the consumer is (strictly) risk averse, $u''(\cdot) < 0$. So, $w - qx^* - L + x^* < w - qx^*$, meaning $x^* < L$. That is, the consumer will not fully insure.

   (b) For what price, if any, will a risk averse consumer not insure at all?

   Solution: The first order condition for $x^* = 0$ is

$$\begin{aligned} \pi u'(w-L)(1-q) + (1-\pi)u'(w)(-q) &\leq 0 \\ \Rightarrow \quad \pi u'(w-L) - \pi u'(w-L)q + (1-\pi)u'(w)(-q) &\leq 0 \\ \Rightarrow \quad q &\geq \frac{\pi u'(w-L)}{\pi u'(w-L) + (1-\pi)u'(w)}. \end{aligned}$$

4. Consider the insurance model discussed in the lecture. Assuming that the price of insurance is fair, how much insurance will a *risk-loving* individual buy? (Please remember to pay attention to the second order condition.)

   Solution: (Note: This solution uses L to denote the possible loss, g to denote a lottery/gamble, a for the probability of loss, and x for the amount of insurance purchase). DM's expected utility maximization problem is

$$\max_{0 \leq x \leq L} U(g(x))$$

$$\Longleftrightarrow \quad \max_{0 \leq x \leq L} au(W - ax - L + x) + (1-a)u(W - ax).$$

   We first find the critical values by solving the first order condition with equality:

$$\begin{aligned} \frac{dU(g(x))}{dx} &= a(1-a)u'\left(W - ax^* - L + x^*\right) - a(1-a)u'\left(W - ax^*\right) \\ &= 0. \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad u'\left(W - ax^* - L + x^*\right) &= u'\left(W - ax^*\right) \\
W - ax^* - L + x^* &= W - ax^* \quad \text{since } u'' > 0 \\
x^* &= L.
\end{aligned}$$

However, as we show below, $x^* = L$ does not satisfy the second order condition:

$$\begin{aligned}
\frac{d^2 U(g(x))}{dx^2} &= a(1-a)^2 u''\left(W - ax^* - L + x^*\right) + a^2(1-a)u''\left(W - ax^*\right) \\
&> 0 \quad \text{since } u'' > 0.
\end{aligned}$$

So, $x^* = L$ is a utility minimizer not a maximizer. The only remaining possibility for a maxima is the left boundary $x^{**} = 0$. We check to see if it satisfies the first order condition for the left boundary:

$$\begin{aligned}
\left.\frac{dU(g(x))}{dx}\right|_{x^{**}=0} &= a(1-a)u'(W-L) - a(1-a)u'(W) \\
&< 0 \quad \text{since } W - L < W \text{ and } u'' > 0.
\end{aligned}$$

So, the first order condition is satisfied and the solution is $x^{**} = 0$.

5. Let L be a lottery with expected value $E[L]$. Let $L' = (p \circ L, (1-p) \circ E[L])$ be a compound lottery that yield L with probability p and $E[L]$ with probability $(1-p)$.

   (a) Show that L second order stochastically dominate L'.

   Solution: Using the linearity of expected utility and the fact that $U(\cdot)$ exhibits risk aversion yields:

   $$U(L') = pU(L) + (1-p)U(E[L]) \geq pU(L) + (1-p)U(L) = U(L).$$

   So L' second order stochastically dominate L since risk-averse decision maker will weakly prefer L'.

6. MWG 6.C.2(a)

   Solution: Since $u(x) = \beta x^2 + \gamma x$, we have

   $$E[u(x)] = \beta E[X^2] + \gamma E[X] = \beta VAR[X] + \beta E[X]^2 + \gamma E[X].$$

7. MWG 6.C.12. Note that part(a) should read $\beta > 0$ if $\rho < 1$, and $\beta < 0$ if $\rho > 1$.

   Solution:

(a) If $u(\cdot)$ exhibits constant relative risk aversion, then for some constant ρ,

$$-\frac{xu''(x)}{u'(x)} = \rho \quad \Longleftrightarrow \quad \frac{u''(x)}{u'(x)} = \frac{-\rho}{x}.$$

Integrating the right equation yields

$$\ln u'(x) = -\rho \ln x + c_1 \quad \Longleftrightarrow \quad u'(x) = e^{c_1} x^{-\rho}.$$

When $\rho \neq 1$, integrating the right equation once more yields

$$u(x) = \left(\frac{e^{c_1}}{1-\rho}\right) x^{1-\rho} + c_2.$$

(b) Integrating

$$u'(x) = e^{c_1} x^{-\rho}$$

when $\rho = 1$ yields

$$u(x) = \left(e^{c_1}\right) \ln x + c_2.$$

(c) The text should have said "$\lim_{\rho\to 1} \frac{x^{1-\rho}-1}{1-\rho}$"

$$\lim_{\rho\to 1} \frac{x^{1-\rho}-1}{1-\rho} = \lim_{\rho\to 1} \frac{-x^{1-\rho}\ln(x)}{-1} = 1.$$

Note that since both the numerator and the denominator of the quotient goes to zero, we have used L'Hopital's rule (differentiating with respect to ρ) to obtain the limit.

8. Consider a risk-averse decision maker who faces following two gambles.

- Gamble A: toss a fair coin once. If head comes up get paid ¥1. If tail comes up, pay ¥1.
- Gamble B: toss a fair coin N times. Get paid ¥1 for each time head comes up and pay ¥1 for each tail that comes up.

Which gamble will the decision maker prefer?

Solution: Let B_n be the version of B in which the coin is tossed n times. B_{n+1} takes the outcome of B_n and adds +1 with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$. Since the term that's being added has mean zero, B_{n+1} is a mean preserving spread of B_n. Therefore,

$$A = B_1 \succ B_2 \succ \cdots \succ B_{N-1} \succ B_N = B.$$

2 General equilibrium

1. Consider an Edgeworth box economy where preferences are given by

$$u_1(x_{11}, x_{21}) = 2\sqrt{x_{11}} + x_{21} \quad \text{and} \quad u_2(x_{12}, x_{22}) = 2\sqrt{x_{12}} + x_{22},$$

   (a) Suppose the initial endowments are $\omega_1 = (4,4)$ and $\omega_2 = (1,1)$. Find all the Pareto optimal allocations.

   Solution: Interior Pareto optimality is characterized by marginal rates of substitution being equal to each other. Thus,

$$MRS_1 = \frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{1}{\sqrt{x_{11}}} = \frac{1}{\sqrt{x_{12}}} = \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}} = MRS_2.$$

   Thus, $x_{11} = x_{12}$. Since $x_{11} + x_{12} = \bar{e}_1 = 5$, interior PO allocations are characterized by $x_{11} = x_{12} = 2.5$. So

$$\text{Interior PO set } = \{((2.5, x_{21}), (2.5, 5 - x_{21})) : 0 \le x_{21} \le 5\}.$$

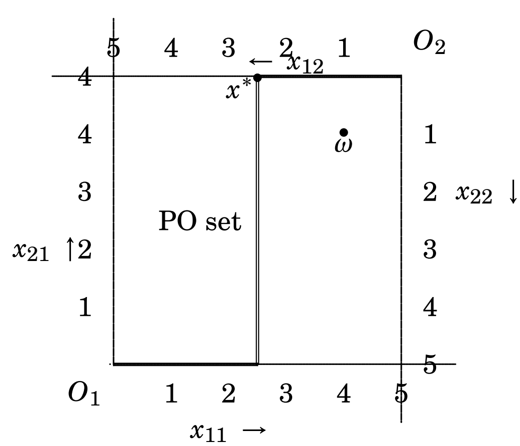
   Note that when $x_{11} < 2.5$ and $x_{21} = 0$ (which means $x_{12} > 2.5$ and $x_{22} = 5$), we have

$$MRS_1 = \frac{1}{\sqrt{x_{11}}} > \frac{1}{\sqrt{x_{12}}} = MRS_2.$$

   So this is a boundary PO allocation. Similarly, when $x_{11} > 2.5$ and $x_{21} = 5$ (which means $x_{12} < 2.5$ and $x_{22} = 0$), we have

$$MRS_1 = \frac{1}{\sqrt{x_{11}}} < \frac{1}{\sqrt{x_{12}}} = MRS_2.$$

   So this is a boundary PO allocations. The set of all the PO allocations are graphed below:



   (b) Find the Walrasian equilibrium.

Solution: To find the Walrasian equilibrium, we first find the demand functions. Using normalization $p_2 = 1$,

$$MRS_1 = \frac{1}{\sqrt{x_{11}}} = \frac{p_1}{1} \implies x_{11} = \frac{1}{p_1^2}$$
$$MRS_2 = \frac{1}{\sqrt{x_{12}}} = \frac{p_1}{1} \implies x_{12} = \frac{1}{p_1^2}.$$

Market clearing condition for Market 1 yields

$$x_{11} + x_{12} = \bar{\omega}_1 \implies \frac{1}{p_1^2} + \frac{1}{p_1^2} = 5 \implies p_1^* = \sqrt{\frac{2}{5}} = 0.6325$$
$$x_{11}^* = \frac{1}{\frac{2}{5}} = 2.5 \quad \text{and} \quad x_{12}^* = \frac{1}{\frac{2}{5}} = 2.5$$

To find equilibrium amount of good 2, we use

$$\begin{aligned}
p_1x_{11} + p_2x_{21} = 4p_1 + 4p_2 &\implies 0.6325(2.5) + x_{21} = 4(0.6325) + 4 \\
&\implies x_{21}^* = 4 + 1.5(0.6325) = 4 + 0.9488 = 4.9488 \\
p_1x_{12} + p_2x_{22} = 1p_1 + 1p_2 &\implies 0.6325(2.5) + x_{22} = 1(0.6325) + 1 \\
&\implies x_{22}^* = 1 - 1.5(0.6325) = 1 - 0.9488 = 0.0512.
\end{aligned}$$

To recap, the equilibrium prices are $p^* = (0.6325, 1)$ and the equilibrium allocation is $\left(x^{1^*}, x^{2^*}\right) = \left((2.5, 4.9488), (2.5, 0.0512)\right)$.

2. Consider an Edgeworth box economy where preferences are given by

$$u_1(x_{11}, x_{21}) = \min\{x_{11}, x_{21}\} \quad \text{and} \quad u_2(x_{12}, x_{22}) = 2x_{12} + x_{22},$$

and the initial endowments are

$$\omega_1 = (3, 2) \quad \text{and} \quad \omega_2 = (1, 2).$$

Using the normalization $p_2 = 1$, find all the Walrasian equilibrium.

Solution: To find consumer 1's Marshallian demand function, we note that the two goods are perfect complements and that an optimal bundle must have $x_{11} = x_{21}$. So, substituting this into her budget equation yields:

$$p_1x_{11} + p_2x_{11} = 3p_1 + 2p_2 \implies x_1(p_1, p_2) = \left(\frac{3p_1 + 2p_2}{p_1 + p_2}, \frac{3p_1 + 2p_2}{p_1 + p_2}\right).$$

To find consumer 2's Marshallian demand function, we note that the two goods are perfect substitutes with the marginal rate of substitution equal to 2. So,

$$x_2(p_1, p_2) = \begin{cases} \left(\frac{p_1 + 2p_2}{p_1}, 0\right) & \text{if } \frac{p_1}{p_2} < 2, \\ \text{any } x_{12},\, x_{22}, \text{ s.t. } p_1x_{12} + p_2x_{22} = p_1 + 2p_2 & \text{if } \frac{p_1}{p_2} = 2 \\ \left(0, \frac{p_1 + 2p_2}{p_2}\right) & \text{if } \frac{p_1}{p_2} > 2. \end{cases}$$

Now, to find the market clearing prices we note the following. Consumer 1 never wants to consumer at the boundary. However, consumer 2 always wants to consume at the boundary unless $\frac{p_1}{p_2} = 2$. Therefore, the only possible candidate for an equilibrium price is $\frac{p_1}{p_2} = 2$.

To see, if $\frac{p_1}{p_2} = 2$ works, we note that at this price consumer 2 will be happy consuming anywhere on her budget line. So, we can clear the market by giving consumer 1 her desired consumption bundle and giving the remainder to consumer 2. That is, Walarsian equilibrium price is $p = (2,1)$, where we have normalized $p_2 = 1$ and the equilibrium allocation is given by

$$x_1 = x_1(2,1) = \left(\frac{8}{3}, \frac{8}{3}\right) \quad \text{and} \quad x_2 = \omega_1 + \omega_2 - x_1 = \left(\frac{4}{3}, \frac{4}{3}\right).$$

3. (2017 MT) Consider a pure exchange economy with 2 consumers and 2 goods, where the consumers' preferences and endowments are given by

$$\begin{aligned} u_1(x_{11}, x_{21}) &= \min\{x_{11}, x_{21}\} \qquad \omega_1 = (6,5) \\ u_2(x_{12}, x_{22}) &= \min\{x_{12}, x_{22}\} \qquad \omega_2 = (6,5). \end{aligned}$$

(a) Using a carefully labeled Edgeworth box diagram, graph the PO Set.

Solution: Pareto set is the thick area between the two dashed lines.

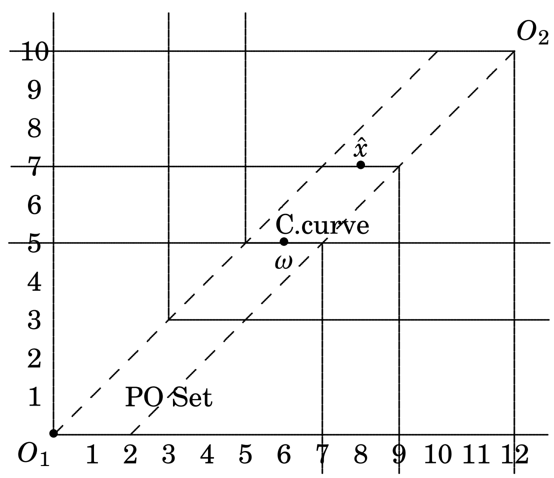


Figure 1: Pareto Set.

(b) Using a carefully labeled Edgeworth box diagram, graph the contract curve.

Solution: Contract curve is the portion of Pareto set where every consumer is doing as well as their endowment. This is the short line segment in the Pareto set that goes through the endowment point.

(c) Can allocation $(\hat{x}_1, \hat{x}_2) = \big((8,7),(4,3)\big)$ be supported as a Walrasian equilibrium with transfers? If so, find the supporting prices and the corresponding transfers. If not, explain.

Solution: Yes, it is possible to support $\hat{x}$ as equilibrium with transfers. Supporting price for $\hat{x}$ is $\hat{p} = (0,1)$. The required transfers are

$$\begin{aligned} T_1 &= \hat{p}\cdot\hat{x}_1 - \hat{p}\cdot\omega_1 = (0,1)\cdot(8,7) - (0,1)\cdot(6,5) = 7-5 = 2 \\ T_2 &= -2. \end{aligned}$$

4. (MWG 15.B.2) Consider an Edgeworth box economy in which consumers have the Cobb-Douglas utility functions $u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha}$ and $u_2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta}$. Consumer i's endowments are $(\omega_{1i}, \omega_{2i}) \gg 0$, for $i = 1,2$. Solve for the equilibrium price ratio and allocation. How do these change with differential change in ω_{11}?.

Solution: Normalize by setting $p_2 = 1$. Then solving the utility maximization problem yields the following demand functions:

$$\begin{aligned} x_{11}(p) &= \frac{\alpha(p_1\omega_{11} + \omega_{21})}{p_1} = \alpha\omega_{11} + \frac{\alpha\omega_{21}}{p_1} \\ x_{21}(p) &= (1-\alpha)(p_1\omega_{11} + \omega_{21}) \\ x_{12}(p) &= \frac{\beta(p_1\omega_{12} + \omega_{22})}{p_1} = \beta\omega_{12} + \frac{\beta\omega_{22}}{p_1} \\ x_{22}(p) &= (1-\beta)(p_1\omega_{12} + \omega_{22}). \end{aligned}$$

To find the equilibrium price, we use the market clearing condition for good 2:

$$\begin{aligned} x_{21}(p^*) + x_{22}(p^*) = \omega_{21} + \omega_{22} &\iff (1-\alpha)(p_1^*\omega_{11} + \omega_{21}) + (1-\beta)(p_1^*\omega_{12} + \omega_{22}) = \omega_{21} + \omega_{22} \\ &\iff p_1^*\big((1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\big) = \alpha\omega_{21} + \beta\omega_{22} \\ &\iff p_1^* = \frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}. \end{aligned}$$

The equilibrium allocations are found by substituting p^* into the demand functions:

$$\begin{aligned} x_{11}^* &= \alpha\omega_{11} + \frac{\alpha\omega_{21}}{p_1^*} = \alpha\omega_{11} + \frac{\alpha\omega_{21}\big((1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\big)}{\alpha\omega_{21} + \beta\omega_{22}} \\ x_{21}^* &= (1-\alpha)(p_1\omega_{11} + \omega_{21}) = (1-\alpha)\omega_{11}\left(\frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}\right) + (1-\alpha)\omega_{21} \\ x_{12}^* &= \beta\omega_{12} + \frac{\beta\omega_{22}}{p_1^*} = \beta\omega_{12} + \frac{\beta\omega_{22}\big((1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\big)}{\alpha\omega_{21} + \beta\omega_{22}} \\ x_{22}^* &= (1-\beta)(p_1\omega_{12} + \omega_{22}) = (1-\beta)\omega_{12}\left(\frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}\right) + (1-\beta)\omega_{21}. \end{aligned}$$

Thus, we have:

$$\frac{\partial p_1^*}{\partial \omega_{11}} = -\frac{(1-\alpha)(\alpha\omega_{21}+\beta\omega_{22})}{\left((1-\alpha)\omega_{11}+(1-\beta)\omega_{12}\right)^2} < 0$$
$$\frac{\partial x_{11}^*}{\partial \omega_{11}} = \alpha + \frac{\alpha\omega_{21}(1-\alpha)}{\alpha\omega_{21}+\beta\omega_{22}} > 0$$
$$\frac{\partial x_{21}^*}{\partial \omega_{11}} > 0 \quad \text{by carefully differentiating}$$
$$\frac{\partial x_{12}^*}{\partial \omega_{11}} > 0 \quad \text{by carefully differentiating}$$
$$\frac{\partial x_{22}^*}{\partial \omega_{11}} = \frac{\partial}{\partial \omega_{11}}\left[\omega_{21}+\omega_{22}-x_{21}^*\right] = -\frac{\partial x_{21}^*}{\partial \omega_{11}} < 0.$$

5. Consider again the economy given in Problem set 5, question 2.

(a) Find all the Walrasian equilibrium using the normalization $p_2 = 1$.
Solution: We solve for the market clearing condition for good 2:

$$x_2(w,1) = \omega_2 + y(w,1) \quad \Longrightarrow \quad \frac{5w + \frac{1}{4w}}{3} = \frac{1}{2w} \quad \Longrightarrow \quad w^* = \frac{1}{2}.$$

Thus, Walrasian equilibrium price is $(w^*, p_2^*) = (\frac{1}{2}, 1)$, and the corresponding Walrasian equilibrium consumption and production plans are $(x_1^*, x_2^*) = (4,1)$ and $(z^*, y^*) = (1,1)$.

(b) Using a single diagram, graph the consumer's utility maximizing-consumption plan and the firm's profit-maximizing production plan for the equilibrium found above.
Solution:

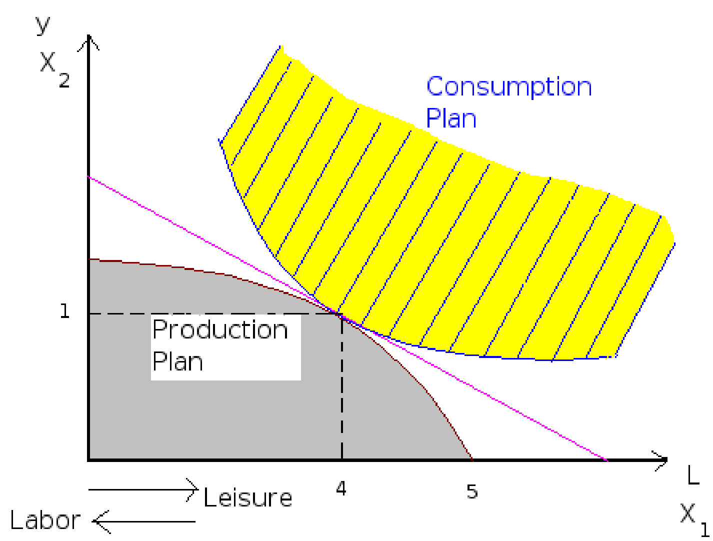


Figure 2: Production and Consumption Plan at equilibrium price.

6. Show that the firms' profit maximizing production plans are homogeneous of degree zero in an Arrow-Debreu economy, while the profit functions are homogeneous of degree one. That is, for all $\alpha > 0$, $y_j(\alpha p) = y_j(p)$ and $\pi_j(\alpha p) = \alpha\pi_j(p)$. Also show that the individuals' demand correspondences are homogeneous of degree zero.

   Solution: For all $\alpha > 0$, we have

$$\max_{y\in Y_J} (\alpha p)\cdot y_j \iff \max_{y\in Y_J} \alpha(p\cdot y_j) \iff \max_{y\in Y_J} p\cdot y_j.$$

   Therefore, $y_j(\alpha p) = y_j(p)$. This then implies that

$$\pi_j(\alpha p) = (\alpha p)\cdot y_j(\alpha p) = (\alpha p)\cdot y_j(p) = \alpha(p\cdot y_j(p)) = \alpha\pi_j(p).$$

   Individual i's budget set when the price vector is αp is

$$\begin{aligned}(\alpha p)\cdot x_i \le (\alpha p)\cdot \omega_i + \pi(\alpha p) &\iff (\alpha p)\cdot x_i \le (\alpha p)\cdot\omega_i + \alpha\pi(p)\\ &\iff p\cdot x_i \le p\cdot\omega_i + \pi(p),\end{aligned}$$

   which is the same budget set as when the price is p. Therefore, $x_i(\alpha p) = x_i(p)$.

7. Suppose function $z:\mathbb{R}^L_{++}\to\mathbb{R}^L$ satisfies Walras' Law (that is, $p\cdot z(p) = 0$ for all $p\in\Delta^\circ$). Show that if $z_\ell(p) = 0$ for any $L-1$ components of $z(p)$, then $z_\ell(p) = 0$ for all ℓ.

   Solution: By Walras' Law, we have

$$p\cdot z(p) = p_1 z_1(p) + \cdots + p_{L-1}z_{L-1}(p) + p_L z_L(p) = 0.$$

   Suppose the markets for $L-1$ goods, say goods $\ell = 1,\ldots,L-1$ clears. Then $p_L z_L(p) = 0$. Since $p_L > 0$, we have $z_L(p) = 0$.

8. Consider an economy with two goods, two consumers and one producer, where the consumers' utility functions and endowments are

$$\begin{aligned} u_1(x_{11},x_{21}) &= \ln x_{11} + \ln x_{21} \qquad & \omega_1 &= (10,10)\\ u_2(x_{12},x_{22}) &= \ln x_{12} + 3\ln x_{22} \qquad & \omega_2 &= (10,10).\end{aligned}$$

   The firm uses good 1 as input to produce good 2. Its production function is $f(z) = z^{\frac{1}{2}}$. The consumers' shares of the firm are given by $\theta_1 = \frac{1}{4}$ and $\theta_2 = \frac{3}{4}$. Find the Walrasian equilibrium.

   Solution: As before, the firm's profit maximization problem is:

$$\max_z\ p_2 z^{\frac{1}{2}} - wz \implies \text{FOC is:}\quad \frac{p_2}{2}z^{-\frac{1}{2}} - w = 0.$$

Solving the FOC and normalizing $p_2 = 1$, we obtain

$$z(p_2, w) = \frac{p_2^2}{4w^2} = \frac{1}{4w^2}$$
$$y(p_2, w) = \left(\frac{p_2^2}{4w^2}\right)^{\frac{1}{2}} = \frac{p_2}{2w} = \frac{1}{2w}$$
$$\pi(p_2, w) = p_2\frac{p_2}{2w} - w\frac{p_2^2}{4w^2} = \frac{p_2^2}{4w} = \frac{1}{4w}.$$

For consumer's demand functions, we first transform the non-standard Cobb-Douglas utility functions we have into standard Cobb-Douglas utility functions.

$$u_1(x_{11}, x_{21}) = \ln x_{11} + \ln x_{21} \sim x_{11}^{\frac{1}{2}} x_{21}^{\frac{1}{2}}$$
$$u_2(x_{12}, x_{22}) = \ln x_{12} + 3\ln x_{22} \sim x_{12}^{\frac{1}{4}} x_{22}^{\frac{3}{4}}.$$

At price $p = (1, w)$, the consumers' wealth are

$$W_1 = (w, 1)\cdot\omega_1 + \theta_1\pi(w, 1) = w(10) + 1(10) + \frac{1}{4}\left(\frac{1}{4w}\right) = 10 + 10w + \frac{1}{16w}$$
$$W_2 = (w, 1)\cdot\omega_2 + \theta_1\pi(w, 1) = w(10) + 1(10) + \frac{3}{4}\left(\frac{1}{4w}\right) = 10 + 10w + \frac{3}{16w}.$$

Therefore the consumers' demand functions are

$$x_{11}(w, 1) = \frac{W_1}{2w}$$
$$x_{21}(w, 1) = \frac{W_1}{2} = \frac{10 + 10w + \frac{1}{16w}}{2} = 5 + 5w + \frac{1}{32w}$$
$$x_{12}(w, 1) = \frac{W_2}{4w}$$
$$x_{22}(w, 1) = \frac{3W_2}{4} = \frac{3(10 + 10w + \frac{3}{16w})}{4} = \frac{15}{2} + \frac{15w}{2} + \frac{9}{64w}.$$

To find the equilibrium price, we solve for market clearing condition for good 2:

$$x_{21}(w, 1) + x_{22}(w, 1) = 5 + 5w + \frac{1}{32w} + \frac{15}{2} + \frac{15w}{2} + \frac{9}{64w} = 20 + \frac{1}{2w} = \bar{\omega}_2 + y(w, 1),$$

which yields

$$\frac{25}{2} + \frac{25w}{2} + \frac{11}{64w} = 20 + \frac{1}{2w}$$
$$800w + 800w^2 + 11 = 1280w + 32$$
$$800w^2 - 480w - 21 = 0$$
$$\implies w^* = \frac{12 + \sqrt{186}}{40} \approx 0.64.$$

The equilibrium allocation $\left(x_{11}^*, x_{21}^*, x_{12}^* x_{22}^*, z^*, y^*\right)$ can be found by substituting $w^* = 0.64$ into the demand and supply functions found above.

9. Consider a 2-goods economy with 2 consumers and 1 firm. The consumers have identical preferences:

$$u_i(x_{1i}, x_{2i}) = x_{1i}^{\frac{1}{2}} x_{2i}^{\frac{1}{2}},$$

while their endowments are $\omega_1 = (10, 0)$ and $\omega_2 = (0, 10)$. The firm uses the first good as input to produce the second good. Its production function is

$$f(z) = z^{\frac{1}{2}},$$

where z denotes the amount of good 1. Let $\theta_1 = \theta_2 = \frac{1}{2}$ be the consumers' shares of the firm.

(a) Find the Walrasian equilibrium (use the normalization $p_2 = 1$).

Solution: Letting $p_2 = 1$, the firm's profit maximization problem is

$$\max_z \; z^{\frac{1}{2}} - p_1 z.$$

The first order condition yields

$$\begin{aligned}
\tfrac{1}{2} z^{-\frac{1}{2}} - p_1 \; &= \; 0 \Longrightarrow z^{-\frac{1}{2}} \; = \; 2p_1 \Longrightarrow z(p) \; = \; \frac{1}{4p_1^2} \\
\Longrightarrow y(p) \; &= \; z(p)^{\frac{1}{2}} \; = \; \frac{1}{2p_1} \\
\Longrightarrow \pi(p) \; &= \; z(p)^{\frac{1}{2}} - p_1 z(p_1) \; = \; \frac{1}{2p_1} - \frac{1}{4p_1} \; = \; \frac{1}{4p_1}.
\end{aligned}$$

Next, consumer i's income is

$$I_1 = p \cdot \omega_1 + \frac{\pi(p)}{2} = 10p_1 + \frac{1}{8p_1} \quad \text{and} \quad I_2 = p \cdot \omega_2 + \frac{\pi(p)}{2} = 10 + \frac{1}{8p_1}$$

Using the well-known formula for Cobb-Douglas utility function, we obtain

$$\begin{aligned}
x_{11}(p) &= \frac{I_1}{2p_1} = \frac{10p_1 + \frac{1}{8p_1}}{2p_1} = 5 + \frac{1}{16p_1^2} \\
x_{21}(p) &= \frac{I_1}{2} = \frac{10p_1 + \frac{1}{8p_1}}{2} = 5p_1 + \frac{1}{16p_1} \\
x_{12}(p) &= \frac{I_2}{2p_1} = \frac{10 + \frac{1}{8p_1}}{2p_1} = \frac{5}{p_1} + \frac{1}{16p_1^2} \\
x_{22}(p) &= \frac{I_2}{2} = \frac{10 + \frac{1}{8p_1}}{2} = 5 + \frac{1}{16p_1}.
\end{aligned}$$

Thus, the market clearing condition for good 2 is

$$\begin{aligned}
x_{11}(p)+x_{12}(p) &= \bar{\omega}_2 + y(p)\\
5p_1 + \frac{1}{16p_1} + 5 + \frac{1}{16p_1} &= 10 + \frac{1}{2p_1}\\
80p_1^2 + 1 + 80p_1 + 1 &= 160p_1 + 8\\
80p_1^2 - 80p_1 - 6 &= 0\\
40p_1^2 - 40p_1 - 3 &= 0
\end{aligned}$$

$$\begin{aligned}
p_1 &= \frac{40 \pm \sqrt{40^2 - 4(40)(-3)}}{2(40)} = \frac{40 \pm \sqrt{1600+480}}{80}\\
&= \frac{40 \pm \sqrt{2080}}{80} = \frac{40 \pm 45.61}{80} = 1.07.
\end{aligned}$$

Substituting $p_1* = 1.07$ in the demand functions yields

$$\begin{aligned}
x_{11}^* &= 5 + \frac{1}{16(1.07)^2} = 5.05 \quad \text{and} \quad x_{21}^* = 5p_1 + \frac{1}{16(1.07)} = 5.41\\
x_{12}^* &= \frac{5}{p_1} + \frac{1}{16(1.07)^2} = 4.73 \quad \text{and} \quad x_{22}^* = 5 + \frac{1}{16(1.07)} = 5.06\\
z^* &= \frac{1}{4(1.07)^2} = 0.22 \quad \text{and} \quad y^* = \frac{1}{2(1.07)} = 0.47.
\end{aligned}$$

(b) Verify that the first fundamental of theorem of welfare holds.

Solution: At the equilibrium,

$$\begin{aligned}
MRS_1 &= \frac{\frac{\partial u_1}{\partial x_{11}}}{\frac{\partial u_1}{\partial x_{21}}} = \frac{x_{21}^*}{x_{11}^*} = \frac{5.41}{5.05} = 1.07\\
MRS_2 &= \frac{\frac{\partial u_2}{\partial x_{12}}}{\frac{\partial u_2}{\partial x_{22}}} = \frac{x_{22}^*}{x_{12}^*} = \frac{5.06}{4.73} = 1.07.
\end{aligned}$$

Thus, the indifference curves of the two consumers are tangent to each other, and we have Pareto optimality.

(c) Which consumer, if any, will be worse off if there was no firm (that is, if this was a pure exchange economy)? Give an interpretation of this result.

Solution: Let the double star ($**$) represent the equilibrium without the firm. Solving the market clearing condition for the second market now yields

$$5p_1^{**} + 5 = 10 \implies p_1^{**} = 1 \implies x_{\ell i}^{**} = 5 \text{ for all } \ell \text{ and } i.$$

Therefore,

$$\begin{aligned}
u_1(x_{11}^{**}, x_{21}^{**}) &= \sqrt{5(5)} = 5 < 5.23 = \sqrt{5.05(5.41)} = u_1(x_{11}^*, x_{21}^*)\\
u_2(x_{12}^{**}, x_{22}^{**}) &= \sqrt{5(5)} = 5 > 4.89\sqrt{4.73(5.06)} = u_1(x_{11}^*, x_{21}^*).
\end{aligned}$$

Note that possibility of production makes the input good (good 1) more valuable (more scarce) and makes the output good (good 2) less valuable (more abundant). Since consumer 1 is the only supplier of good 1 while consumer 2 is the only supplier of good 2, it makes sense that consumer is better off with the production while consumer 2 is not.

10. Consider a 2-goods economy with 2 consumers and 1 firm. The consumers have identical preferences and endowments:

$$u_i(x_{1i}, x_{2i}) = x_{1i} + \ln x_{2i} \quad \text{and} \quad \omega_i = (10, 0), \qquad i = 1, 2.$$

The firm uses the first good as input to produce the second good. Its production function is

$$f(v) = v^{\frac{1}{2}},$$

where v denotes the amount of good 1 the firm uses as input (we are using v rather than the typical z to denote the input good since $z(p)$ is used for the market excess demand function). Let θ_i denote consumer i's share of the firm.

(a) Normalizing $p_2 = 1$, find the market excess demand function.

Solution: Letting $p_2 = 1$, the firm's profit maximization problem is

$$\max_{v} \; v^{\frac{1}{2}} - p_1 v.$$

The first order condition yields

$$\begin{aligned}
\tfrac{1}{2}v^{-\frac{1}{2}} - p_1 &= 0 \Longrightarrow v^{-\frac{1}{2}} = 2p_1 \Longrightarrow v(p) = \frac{1}{4p_1^2} \\
\Longrightarrow y(p) &= v(p_1)^{\frac{1}{2}} = \frac{1}{2p_1} \\
\Longrightarrow \pi(p) &= v(p_1)^{\frac{1}{2}} - p_1 v(p_1) = \frac{1}{2p_1} - \frac{1}{4p_1} = \frac{1}{4p_1}.
\end{aligned}$$

Next, consumer i's utility maximization problem is

$$\max_{x_{1i}, x_{2i}} \; x_{1i} + \ln x_{2i} \quad \text{s.t.} \quad p_1 x_{1i} + x_{2i} = 10p_1 + \frac{\theta_i}{4p_1}.$$

The solution must satisfy:

$$\begin{aligned}
\frac{\frac{\partial u_i}{\partial x_{1i}}}{\frac{\partial u_i}{\partial x_{2i}}} &= \frac{1}{\frac{1}{x_{2i}}} = \frac{p_1}{1} \\
\Longrightarrow x_{2i}(p) &= p_1 \\
\Longrightarrow x_{1i}(p) &= \frac{10p_1 + \frac{\theta_i}{4p_1} - p_1}{p_1} = 9 + \frac{\theta_i}{4p_1^2}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
z_1(p) &= x_{11}(p)+x_{12}(p)+v(p)-\bar{\omega}_1 \;=\; 18+\frac{\theta_1+\theta_2}{4p_1^2}+\frac{1}{4p_1^2}-20 \\
&= \frac{1}{2p_1^2}-2 \\
z_2(p) &= x_{21}(p)+x_{22}(p)-y(p)-\bar{\omega}_2 \;=\; 2p_1-\frac{1}{2p_1}.
\end{aligned}
$$

(b) Find the Walrasian equilibrium price, allocations, and production plans. How do the equilibrium price and allocations depend on θ_1? Given an intuitive explanation for these results in words.

Solution: Market clearing condition for the second good yields:

$$
z_2(p)=0 \iff 2p_1 \;=\; \frac{1}{2p_1} \iff p_1^2 \;=\; \frac{1}{4} \implies p_1^* = \frac{1}{2}.
$$

Thus, the equilibrium price is $p^* = (\frac{1}{2}, 1)$ and the equilibrium allocations are:

$$
\begin{aligned}
x_1^* &= \left(9+\theta_1,\ \frac{1}{2}\right) \\
x_2^* &= \left(9+(1-\theta_1),\ \frac{1}{2}\right) \\
(v^*, y^*) &= (1,\ 1).
\end{aligned}
$$

As seen above, θ_1 does not affect the equilibrium price or the equilibrium allocation for the second good. This is because the preferences are quasi-linear with respect to the first good, which means that any wealth effects are absorbed by the demand for the first good. That is, changes in wealth do not affect the demand for good 2. Thus, since changes in θ_1 only affects the wealth of the consumers, its effects are absorbed by the first good. In particular, since increase in θ_1 increases consumer 1's wealth and decreases consumer 2's wealth, x_{11}^* is increasing in θ_1 and x_{12}^* is decreasing in θ_1.

(c) Explicitly calculate the index of the equilibrium, and determine whether this is a regular economy.

Solution: Since this is a two goods economy, $\hat{z}(p) = z_1(p_1)$. We have

$$
\frac{\partial z_1}{\partial p_1}\Big|_p^* \;=\; -\frac{1}{p_1^3}\Big|_{p^*} \;=\; -\frac{1}{8} \;\neq\; 0,
$$

so the economy is regular. The index of the equilibrium is

$$
\text{index}(p^*) \;=\; (-1)^{2-1}\text{sign}\left(\frac{\partial z_1}{\partial p_1}\Big|_{p^*}\right) \;=\; (-1)(-1) \;=\; +1.
$$

11. Consider a pure exchange economy with 2 consumers and two goods. Suppose initially, the consumers' endowment allocation is $\omega' = (\omega'_1, \omega'_2) = \big((8,0),(0,4)\big)$, and the equilibrium price is $p' = (1,1)$. However, there is a shock to the economy that causes the endowment allocation to change to $\omega'' = (\omega''_1, \omega''_2) = \big((2,6),(0,2)\big)$. Can you predict what will happen to the equilibrium price ratio? That is, will the equilibrium price ratio $\frac{p''_1}{p''_2}$ be larger, smaller, or same as $\frac{p'_1}{p'_2}$? Assume that the consumers' preferences are continuous, strongly monotone, and strictly convex and that the preferences are not affected by the shock.

Solution: Let x'_1 be the consumer 1's bundle in the original equilibrium. As seen in the graph, x'_1 must necessarily lie to the right of ω''_1, which means consumer 1's indifference curve going through her new endowment ω''_i must be stepper than $\frac{p'_1}{p'_2} = 1$. So the new equilibrium cannot have a budget line flatter than the orginal budget line. That is the new budget line in equilibrium must be steeper, or $\frac{p''_1}{p''_2} > \frac{p'_1}{p'_2}$.

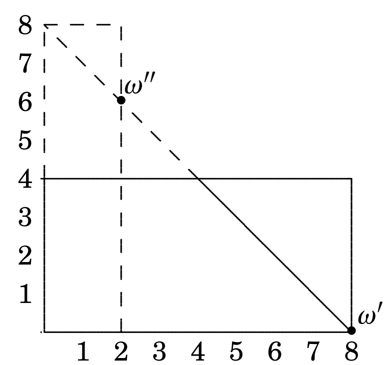


Figure 3: Question 4.